

Rigid C^* -tensor categories of bimodules over interpolated free group factors

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Abstract

Given a countably generated rigid C^* -tensor category \mathcal{C} , we construct a planar algebra P_\bullet whose category of projections \mathbf{Pro} is equivalent to \mathcal{C} . From P_\bullet , we use methods of Guionnet-Jones-Shlyakhtenko-Walker to construct a rigid C^* -tensor category \mathbf{Bim} whose objects are bifinite bimodules over an interpolated free group factor, and we show \mathbf{Bim} is equivalent to \mathbf{Pro} . We use these constructions to show \mathcal{C} is equivalent to a category of bifinite bimodules over $L(\mathbb{F}_\infty)$.

1 Introduction

Jones initiated the modern theory of subfactors in his breakthrough paper [Jon83] in which he classified the possible values for the index of a II_1 subfactor to the range $\{4\cos^2(\pi/n) | n \geq 3\} \cup [4, \infty]$, and he found a subfactor of the hyperfinite II_1 -factor R for each allowed index. In that paper, he asked the following question, which is still unanswered:

Question 1.1. *For which values of the index is there an irreducible subfactor of R ?*

A finite index subfactor $N \subset M$ is studied by analyzing its standard invariant, i.e., two rigid C^* -tensor categories of $N - N$ and $M - M$ bimodules and the module categories of $N - M$ and $M - N$ bimodules which arise from the Jones tower. The standard invariant has been axiomatized in three similar ways, each emphasizing slightly different structure: Ocneanu's paragroups [Ocn88, EK98], Popa's λ -lattices [Pop95], and Jones' planar algebras [Jon99]; we use the planar algebra formalism in the sequel. The above question can be refined:

Question 1.2. *For which subfactor planar algebras P_\bullet is there a subfactor of R whose planar algebra is P_\bullet ?*

In [Pop93, Pop95, Pop02], Popa started with a λ -lattice $A_{\bullet,\bullet} = (A_{i,j})$ and constructed a II_1 -subfactor whose standard invariant was $A_{\bullet,\bullet}$. Hence for each subfactor planar algebra P_\bullet , there is some subfactor whose planar algebra is P_\bullet . Using

this theorem, Popa gave a positive answer to Question 1.2 for (strongly) amenable subfactor planar algebras [Pop94].

In [PS03], Popa and Shlyakhtenko were able to identify the factors in certain cases of Popa's reconstruction theorems. Using this, they gave a positive answer to Question 1.2 for $L(\mathbb{F}_\infty)$, i.e., every subfactor planar algebra arises as the standard invariant of some subfactor $N \subset M$ such that N, M are both isomorphic to $L(\mathbb{F}_\infty)$. This theorem was reproduced by Hartglass [Har12] using the reconstruction results of Guionnet-Jones-Shlyakhtenko-Walker (GJSW) [GJS10, JSW10, GJS11] which produce subfactors of interpolated free group factors.

It is natural to extend these questions to rigid C^* -tensor categories, i.e.,

Question 1.3. *For which rigid C^* -tensor categories \mathcal{C} is there a category \mathcal{C}_{bim} of bifinite bimodules over R such that \mathcal{C}_{bim} is equivalent to \mathcal{C} ?*

As in the subfactor case, Hayashi and Yamagami gave a positive result for amenable rigid C^* -tensor categories [HY00] (amenability for C^* -tensor categories was first studied by Hiai and Izumi [HI98]). Moreover, given a rigid C^* -tensor category \mathcal{C} , Yamagami constructed a category of bifinite bimodules \mathcal{C}_{bim} over an amalgamated free product II_1 -factor such that \mathcal{C}_{bim} is equivalent to \mathcal{C} [Yam03]. However, one can show these factors have property Γ , so they are not interpolated free group factors.

In this paper, we give a result analogous to Popa and Shlyakhtenko's results for $L(\mathbb{F}_\infty)$ for countably generated rigid C^* -tensor categories.

Theorem 1.4. *Every countably generated rigid C^* -tensor category can be realized as a category of bifinite bimodules over $L(\mathbb{F}_\infty)$.*

On the other end of the spectrum, one should also note that there has been interesting work on rigid C^* -tensor categories of bimodules over II_1 -factors by Vaes, Falguières, and Raum [FV11, FR12]. Given a rigid C^* -tensor category \mathcal{C} which is either $\text{Rep}(G)$ for G a compact quantum group [FV11] or a unitary fusion category [FR12], they construct a II_1 -factor M whose category $\text{Bim}(M)$ of bifinite bimodules is exactly \mathcal{C} (up to equivalence). Their results can be interpreted as rigidity results in contrast to the universality of $L(\mathbb{F}_\infty)$ to rigid C^* -tensor categories.

There are three steps to the proof of Theorem 1.4.

- (1) Given a countably generated C^* -tensor category \mathcal{C} , we get a fantastic planar algebra P_\bullet such that the C^* -tensor category Pro of projections of P_\bullet is equivalent to \mathcal{C} .

Recall from [MP12] that a fantastic planar algebra is an unshaded, spherical, evaluable C^* -planar algebra.

This step is well known to experts; we give most of the details in Section 2.

- (2) Given a fantastic planar algebra P_\bullet , we construct a II_1 -factor M and two rigid C^* -tensor categories of bifinite bimodules over M :

- Bim , built entirely from P_\bullet and obviously equivalent to Pro , and

- CF, formed using Connes' fusion and linear operators.

These categories are defined in Definitions 3.24 and 3.26. We then show $\mathbf{Bim} \simeq \mathbf{CF}$ in Theorem 3.27.

We use results of GJSW to accomplish this step in Section 3. Along the way, we adapt Brothier's treatment [Bro12] of GJSW results [GJS10, JSW10] for unshaded planar algebras.

- (3) We show $M \cong L(\mathbb{F}_\infty)$.

This last step is similar to results of GJS [GJS11] and Hartglass [Har12].

One can use similar analysis as in the proof of Theorem 1.4 to prove the following theorem:

Theorem 1.5. *Suppose that in addition, \mathcal{C} has finitely many isomorphism classes of simple objects, i.e., \mathcal{C} is a unitary fusion category. Picking a self-dual object $X \in \mathcal{C}$ which generates \mathcal{C} and such that $\dim(X) > 1$, then \mathcal{C} can be realized as a category of bifinite bimodules over $L(\mathbb{F}_t)$ with*

$$t = 1 + \dim(\mathcal{C})(\dim(X) - 1),$$

where $\dim(\mathcal{C})$ is the Frobenius-Perron dimension of \mathcal{C} .

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2 Tensor categories and planar algebras

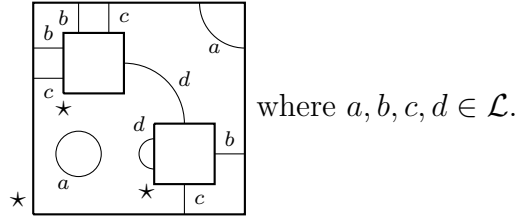
We briefly recall how to go back and forth between rigid C^* -tensor categories and fantastic planar algebras. The contents of this subsection are well known to experts. Our treatment follows [MPS10, Jon11, Yam12, Gho11, Yam03, Müg10].

Notation 2.1. Categories will be denoted with the sans-serif font $\mathbf{ABC} \dots$. We write $X \in \mathcal{C}$ to mean X is an object in \mathcal{C} , and we write $\mathcal{C}(X \rightarrow Y)$ or $\mathcal{C}(X, Y)$ for the set of morphisms from X to Y in \mathcal{C} .

2.1 Fantastic planar algebras and principal graphs

We briefly recall the definition of an unshaded fantastic planar algebra where the strings are labelled. For more details, see [Jon11].

Definition 2.2. Given a set \mathcal{L} , the planar operad with string labels \mathcal{L} , denoted $\mathbb{P}_{\mathcal{L}}$, is the set of all planar tangles whose strings are labelled by elements of \mathcal{L} , e.g.,



A planar tangle T consists of the following data:

- (1) a smooth disk $D_0(T) \subset \mathbb{R}^2$,
- (2) Finitely many disjoint smooth disks in the interior of $D_0(T)$, which we call $D_1(T), \dots, D_s(T)$ (s may be zero),
- (3) Finitely many disjoint smooth arcs in $D_0(T)$ called the strings of T which do not meet the interior of any $D_1(T), \dots, D_s(T)$. The boundary points of a string of T (if it has any) lie in the boundaries of the $D_i(T)$, and they meet these boundaries transversally (if they meet the boundaries at all).

The boundaries of the strings divide the boundaries of the disks into intervals. For each disk, there is a distinguished interval denoted \star . The intervals of $D_i(T)$ are divided by the marked points of $D_i(T)$, i.e., the points at which the strings meet the boundary of $D_i(T)$. Starting at \star , the marked points on each disk are numbered clockwise.

Each string is labelled by an element from \mathcal{L} , which induces a labeling on the marked points of the $D_i(T)$. Reading clockwise around the boundary of $D_i(T)$ starting at \star , we get a word $\alpha_i \in \Lambda$, the set of all finite words on \mathcal{L} . We call $D_i(T)$ an α_i -disk, and we call such a planar tangle T a planar α_0 -tangle.

If we have two planar tangles S, T satisfying the following boundary conditions:

- some internal disk $D_i(S)$ agrees with $D_0(T)$,
- the marked points of $D_i(S)$ agree with the marked points of $D_0(T)$,
- the distinguished interval of $D_i(S)$ agrees with the distinguished interval of T , and
- the label from \mathcal{L} of each marked point of $D_i(S)$ agrees with the label of each marked point from $D_0(T)$,

then we may compose S and T to get the planar tangle $S \circ_i T$ by taking S union the interior of $D_0(T)$, removing the boundary of $D_i(S)$, and smoothing the strings.

Remark 2.3. When we draw a planar tangle, we will often suppress the external disk, which is assumed to be a large rectangle. If we draw the disks as rectangles, it is always assumed the \star is in the lower left corner. Finally, if $a_1, \dots, a_k \in \mathcal{L}$ and $\alpha = a_1 \dots a_k \in \Lambda$, we draw one string labelled α rather than k parallel strings labelled a_1, \dots, a_k where we always read the strands from left to right and top to bottom.

For each word $\alpha = a_1 \dots a_n \in \Lambda$, we write $\bar{\alpha} = a_n \dots a_1$ for the word in the reverse order.

Definition 2.4. A planar algebra P_\bullet with string labels \mathcal{L} is

- a collection of vector spaces $(P_\alpha)_{\alpha \in \Lambda}$ (recall Λ is the set of finite words on \mathcal{L})
- an action of planar tangles by multilinear maps, i.e., for each planar α_0 -tangle T , whose disks $D_i(T)$ are α_i -disks, there is a multilinear map

$$Z_T: \prod_{i=1}^s P_{\alpha_i} \rightarrow P_{\alpha_0}$$

satisfying the following axioms:

Isotopy: If θ is an orientation preserving diffeomorphism of \mathbb{R}^2 , then $Z_{\theta(T)} = Z_T$.

Naturality: For S, T composable tangles, $Z(S \circ_i T) = Z(S) \circ_i Z(T)$, where the composition on the right hand side is the composition of multilinear maps.

Moreover, P_\bullet is called a fantastic planar algebra if P_\bullet is

- evaluable, i.e., $\dim(P_\alpha) < \infty$ for all $\alpha \in \Lambda$ and $P_\emptyset \cong \mathbb{C}$ via the map that sends the empty diagram to $1 \in \mathbb{C}$. Hence, by naturality, to each $a \in \mathcal{L}$, there is a scalar $\delta_a \in \mathbb{C}$ such that any labelled diagram containing a closed loop labelled a is equal to the same diagram without the closed loop multiplied by δ_a . We use the notation $\delta_\alpha = \delta_{a_1} \dots \delta_{a_n}$ if $\alpha = a_1 \dots a_n$.
- involutive, i.e., for each $\alpha \in \Lambda$, there is a map $*$: $P_\alpha \rightarrow P_{\bar{\alpha}}$ with $* \circ * = \text{id}$ which is compatible with the reflection of tangles, i.e., if T is a planar tangle labelled by $\alpha_1, \dots, \alpha_s$, then

$$T(\alpha_1, \dots, \alpha_n)^* = T^*(\alpha_1^*, \dots, \alpha_n^*)$$

where T^* is the reflection of T .

- spherical, i.e., for all $\alpha \in \Lambda$ and all $x \in P_{\alpha\bar{\alpha}}$, we have

$$\text{tr}(x) = \begin{array}{c} \alpha \\ \square \\ \alpha \end{array} \bar{\alpha} = \bar{\alpha} \begin{array}{c} \alpha \\ \square \\ \alpha \end{array}.$$

- positive, i.e., for every $\alpha \in \Lambda$, the map $\langle \cdot, \cdot \rangle : P_\alpha \times P_\alpha \rightarrow P_\emptyset \cong \mathbb{C}$ given by

$$\langle x, y \rangle = \boxed{x} \overset{\alpha}{-} \boxed{y^*}$$

is a positive definite inner product. Hence for all $a \in \mathcal{L}$, $\delta_a > 0$.

Notation 2.5. For each $\alpha, \beta \in \Lambda$, we write $P_{\alpha \rightarrow \beta}$ to denote the box space $P_{\alpha\bar{\beta}}$ of elements of the form

$$\boxed{x} \overset{\alpha}{\underset{\beta}{|}}.$$

Remark 2.6. Note that for each $\alpha \in \Lambda$, the multiplication tangle

$$\begin{array}{c} \alpha \\ | \\ \boxed{} \\ | \\ \alpha \\ | \\ \boxed{} \\ | \\ \alpha \end{array}$$

makes $P_{\alpha \rightarrow \alpha}$ into an associative algebra. If P_\bullet is a fantastic planar algebra, then the multiplication tangle makes $P_{\alpha \rightarrow \alpha}$ a finite dimensional C^* -algebra.

Definition 2.7. Suppose P_\bullet is a fantastic planar algebra. A projection in P_\bullet is an element $p \in P_{\alpha \rightarrow \alpha}$ satisfying $p = p^2 = p^*$ where the multiplication is as in Remark 2.6. A projection $p \in P_{\alpha \rightarrow \alpha}$ is called simple if it is a minimal projection in $P_{\alpha \rightarrow \alpha}$. Since $P_{\alpha \rightarrow \alpha}$ is a finite dimensional C^* -algebra, every projection is the sum of finitely many simple projections. This property is called semi-simplicity.

Given a projection $p \in P_{\alpha \rightarrow \alpha}$, the dual projection $\bar{p} \in P_{\bar{\alpha} \rightarrow \bar{\alpha}}$ is obtained by

$$\bar{p} = \left. \begin{array}{c} \bar{\alpha} \\ | \\ \alpha \\ | \\ \boxed{p} \\ | \\ \alpha \end{array} \right|_{\bar{\alpha}} = \left. \begin{array}{c} \alpha \\ | \\ \boxed{p} \\ | \\ \alpha \end{array} \right|_{\bar{\alpha}}.$$

Projections $p \in P_{\alpha \rightarrow \alpha}$ and $q \in P_{\beta \rightarrow \beta}$ are isomorphic or equivalent, denoted $p \simeq q$, if there is a $u \in P_{\alpha \rightarrow \beta}$ such that

$$u^*u = \begin{array}{c} \alpha \\ | \\ \boxed{u} \\ | \\ \beta \\ | \\ \boxed{u^*} \\ | \\ \alpha \end{array} = p \text{ and } uu^* = \begin{array}{c} \beta \\ | \\ \boxed{u^*} \\ | \\ \alpha \\ | \\ \boxed{u} \\ | \\ \beta \end{array} = q.$$

Given a projection $p \in P_{\alpha \rightarrow \alpha}$ and a $\beta \in \Lambda$, we can form the projection

$$p \otimes \text{id}_\beta = \left. \begin{array}{c} \alpha \\ | \\ \boxed{p} \\ | \\ \alpha \end{array} \right|_{\beta} \in P_{\alpha\beta \rightarrow \alpha\beta}.$$

The principal graph of P_\bullet with respect to $\beta \in \Lambda$, denoted Γ_β , is the graph whose vertices are the isomorphism classes of simple projections in P_\bullet , and if $p \in P_{\alpha \rightarrow \alpha}$ and $q \in P_{\alpha\beta \rightarrow \alpha\beta}$ are simple projections, then the vertices $[p]$ and $[q]$ are connected by $\dim(qP_{\alpha\beta \rightarrow \alpha\beta}(p \otimes \text{id}_\beta))$ edges.

The principal graph of P_\bullet , denoted Γ , is the push out of the Γ_b for $b \in \mathcal{L}$ over the isomorphism classes of simple projections, i.e., the vertices are the same as before, and the edge set is the union of the edge sets of the Γ_b for $b \in \mathcal{L}$.

Since P_\bullet is fantastic and \mathcal{L} is countable, Γ has countably many vertices, although it may not be locally finite. However, Γ_b is always locally finite for $b \in \mathcal{L}$.

Given a vertex $[p]$ of Γ , the number $\text{tr}(p)$ is independent of the choice of representative of $[p]$. The vector $(\text{tr}(p))_{[p] \in V(\Gamma)}$ defines a Frobenius-Perron weight vector on the vertices of Γ satisfying the following equation for each $b \in \mathcal{L}$:

$$\delta_b \text{tr}(p) = \sum_{[q] \in V(\Gamma_b)} n_{[p],[q]}^b \text{tr}(q)$$

where $n_{[p],[q]}^b$ is the number of edges connecting $[p]$ and $[q]$ in Γ_b .

2.2 Rigid C^* -tensor categories and fusion graphs

We briefly recall the definition of a rigid C^* -tensor category.

Definition 2.8. A rigid C^* -tensor category is a pivotal, spherical, positive/unitary, rigid, semisimple, linear (**Vect**-enriched) monoidal category such that $\text{End}(1) \cong \mathbb{C}$.

We now unravel this definition and state many properties that follow. The interested reader should see [Yam03, Müg10] for more details. As we go through the properties, we will also go through the well-known graphical calculus used for strict tensor categories. We will immediately see that we get a fantastic planar algebra from a rigid C^* -tensor category.

We start with an abelian category \mathcal{C} together with

- a bifunctor $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ which is associative up to a natural isomorphism (the pentagon axiom is satisfied), and
- a unit object $1 \in \mathcal{C}$ which is a left and right identity for \otimes up to natural isomorphism (the triangle identity is satisfied).

Remark 2.9. Recall that a tensor category is called strict if the above natural isomorphisms are identities, i.e., for each $X, Y, Z \in \mathcal{C}$, we have

$$(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z) \text{ and} \\ 1 \otimes X = X = X \otimes 1.$$

Since each tensor category is equivalent to a strict tensor category by Theorem 7.2.1 in [ML98], our tensor categories will be assumed to be strict unless otherwise stated. Note that even with all the properties we want, we can still restrict our attention to strict categories.

First, since \mathbf{C} is \mathbf{Vect} -enriched, for each $X, Y \in \mathbf{C}$, $\mathbf{C}(X \rightarrow Y)$ is a finite dimensional complex vector space. The morphisms in \mathbf{C} are drawn as boxes with strings emanating from the top and bottom. The strings are labelled by the objects, and the diagram is read from top to bottom. For example,

$$f = \begin{array}{c} |X \\ \boxed{f} \\ |Y \end{array} \in \mathbf{C}(X \rightarrow Y),$$

and the identity morphism id_X is denoted by the horizontal strand labelled X . We compose morphisms by vertical concatenation

$$\begin{array}{c} |X \\ \boxed{f \circ g} \\ |Z \end{array} = \begin{array}{c} |X \\ \boxed{g} \\ |Y \\ \boxed{f} \\ |Z \end{array} \in \mathbf{C}(X \rightarrow Z),$$

and we tensor morphisms by horizontal concatenation:

$$\begin{array}{c} |X_1 \otimes Y_1 \\ \boxed{f_1 \otimes f_2} \\ |X_2 \otimes Y_2 \end{array} = \begin{array}{c} |X_1 \\ \boxed{f_1} \\ |Y_1 \end{array} \begin{array}{c} |X_2 \\ \boxed{f_2} \\ |Y_2 \end{array} \in \mathbf{C}(X_1 \otimes X_2 \rightarrow Y_1 \otimes Y_2).$$

Since \mathbf{C} is rigid, for each $X \in \mathbf{C}$, there is a *dual* or *conjugate* $\bar{X} \in \mathbf{C}$, and there is a natural isomorphism $\bar{\bar{X}} \cong X$. Along with the dual object, we have an *evaluation map* $\text{ev}_X: \bar{X} \otimes X \rightarrow 1$ and a *coevaluation map* $\text{coev}_X: 1 \rightarrow X \otimes \bar{X}$ such that the diagram

$$\begin{array}{ccc} & X \otimes \bar{X} \otimes X & \\ \text{coev}_X \otimes 1 \nearrow & & \searrow 1 \otimes \text{ev}_X \\ X & \xrightarrow{\text{id}_X} & X \\ 1 \otimes \text{coev}_{\bar{X}} \searrow & & \nearrow \text{ev}_{\bar{X}} \otimes 1 \\ & X \otimes \bar{X} \otimes X & \end{array}$$

commutes. The evaluation is denoted by a cap, and we draw a cup for the coevaluation:

$$\text{ev}_X = \begin{array}{c} \bar{X} \quad X \\ \text{cap} \\ 1 \end{array} \quad \text{and} \quad \text{coev}_X = \begin{array}{c} 1 \\ \text{cup} \\ X \quad \bar{X} \end{array}.$$

The diagram above commuting is sometimes referred to as the *zig-zag relation*, since it is the straightening of the kinked string:

$$\begin{array}{c} |X \\ \text{cup} \\ X \quad \bar{X} \\ \text{cap} \\ X \end{array} = \begin{array}{c} |X \\ \text{cap} \\ X \quad \bar{X} \\ \text{cup} \\ X \end{array}.$$

In general, we don't draw a string connected to the trivial object $1 \in \mathbb{C}$. For each $X, Y \in \mathbb{C}$, $\overline{X} \otimes \overline{Y}$ is naturally isomorphic to $\overline{Y} \otimes \overline{X}$, and the diagram

$$\begin{array}{ccc} \overline{X} \otimes \overline{Y} \otimes (X \otimes Y) & \xrightarrow{\text{ev}_{X \otimes Y}} & 1 \\ \downarrow & & \uparrow \text{ev}_Y \\ \overline{Y} \otimes \overline{X} \otimes X \otimes Y & \xrightarrow{1 \otimes \text{ev}_X \otimes 1} & \overline{Y} \otimes Y \end{array}$$

commutes, and similarly for the coev's. This diagram just means that we can write one cap labelled $X \otimes Y$ and its dual instead of two separate caps labelled X and Y and their duals:

$$\overline{Y} \overline{X} \text{ (cap)}^{X \otimes Y} = \overline{X \otimes Y} \text{ (cap)},$$

and similarly for the cups.

The *pivotality* axiom in \mathbb{C} requires that for all $f \in \mathbb{C}(X \rightarrow Y)$,

$$(\text{ev}_Y \otimes \text{id}_{\overline{X}}) \circ (\text{id}_{\overline{Y}} \otimes f \otimes \text{id}_{\overline{X}}) \circ (\text{id}_{\overline{Y}} \otimes \text{coev}_X) = (\text{id}_X \otimes \text{ev}_Y) \circ (\text{id}_{\overline{Y}} \otimes f \otimes \text{id}_{\overline{X}}) \circ (\text{coev}_{\overline{X}} \otimes \text{id}_{\overline{Y}}).$$

The equation above has an elegant representation in diagrams:

$$\overline{Y} \left| \begin{array}{c} \text{X} \\ \boxed{f} \\ \text{Y} \end{array} \right|_{\overline{X}} = \left| \begin{array}{c} \text{X} \\ \boxed{f} \\ \text{Y} \end{array} \right|_{\overline{X}} \overline{Y}.$$

The evaluations and coevaluations together with pivotality allow us to define a *left* and *right trace* on $\text{End}_{\mathbb{C}}(X)$:

$$\begin{aligned} \text{tr}_L(f) &= \text{ev}_X \circ (\text{id}_{\overline{X}} \otimes f) \circ \text{coev}_{\overline{X}} = \overline{X} \left| \begin{array}{c} \text{X} \\ \boxed{f} \\ \text{X} \end{array} \right| \in \text{End}(1) \cong \mathbb{C} \text{ and} \\ \text{tr}_R(f) &= \text{ev}_{\overline{X}} \circ (f \otimes \text{id}_{\overline{X}}) \circ \text{coev}_X = \left| \begin{array}{c} \text{X} \\ \boxed{f} \\ \text{X} \end{array} \right| \overline{X} \in \text{End}(1) \cong \mathbb{C}. \end{aligned}$$

Similarly, for each $X \in \mathbb{C}$, there are numbers d_X^L and d_X^R which are the left and right traces of the identity morphism respectively, and $d_X^L = d_{\overline{X}}^R$ and $d_X^R = d_{\overline{X}}^L$.

Sphericity means that these two traces are equal, and we denote the common number by $\text{tr}(f)$. The sphericity allows us to perform isotopy on closed diagrams as if they were drawn on a sphere. Hence $\dim(X) := d_X^L = d_X^R$ and $\dim(X) = \dim(\overline{X})$ for all $X \in \mathbb{C}$.

The *positivity* or *unitarity* of \mathbb{C} means there is a contravariant functor $*$: $\mathbb{C} \rightarrow \mathbb{C}$ which is the identity on all objects, and on morphisms, it is anti-linear, involutive ($* \circ * = \text{id}_{\mathbb{C}}$), monoidal ($(f \circ g)^* = g^* \circ f^*$ for composable f, g), and positive ($f^* \circ f = 0$ implies $f = 0$). We require $*$ to be compatible with the evaluations and coevaluations, i.e., for all $X \in \mathbb{C}$, $\text{coev}_X = \text{ev}_{\overline{X}}^*$. On diagrams, we perform $*$

by reflecting the diagram, keeping the labels on the strings, and placing a $*$ on all morphisms.

For all $X, Y \in \mathbf{C}$, we now have that $\mathbf{C}(X \rightarrow Y)$ is a Banach space with positive definite inner product

$$\langle f, g \rangle = \text{tr}(g^* f) = \left(\begin{array}{c} X \\ \boxed{f} \\ Y \\ \boxed{g^*} \\ X \end{array} \right) \overline{X} .$$

The inner product makes $\text{End}_{\mathbf{C}}(X)$ a finite dimensional C^* -algebra, so in particular, all projections are sums of finitely many simple projections, and \mathbf{C} is *semi-simple*, i.e., every exact sequence in \mathbf{C} splits. This also means that any object in \mathbf{C} can be written as a finite direct sum of simple objects. Recall that $X \in \mathbf{C}$ is *simple* if $\dim(\text{End}_{\mathbf{C}}(X)) = 1$. Thus if X, Y are non-isomorphic simple objects, $\mathbf{C}(X, Y) = (0)$. This means that for each simple $X, Y, Z \in \mathbf{C}$, there are non-negative integers $N_{X,Y}^Z$ such that $X \otimes Y = \bigoplus_{Z \in \mathbf{C}} N_{X,Y}^Z Z$, i.e.,

$$N_{X,Y}^Z = \dim(\mathbf{C}(X \otimes Y \rightarrow Z)).$$

Moreover, we have *Frobenius reciprocity*, i.e., for each $X, Y, Z \in \mathbf{C}$, there are natural isomorphisms

$$\mathbf{C}(X \otimes Y \rightarrow Z) \cong \mathbf{C}(X \rightarrow Z \otimes \overline{Y}) \cong \mathbf{C}(Y \rightarrow \overline{X} \otimes Z)$$

which are implemented by the evaluation and coevaluation maps:

$$\begin{array}{c} X \mid Y \\ \boxed{f} \\ \mid Z \end{array} \leftrightarrow \begin{array}{c} X \mid Y \\ \boxed{f} \\ \mid Z \end{array} \begin{array}{c} \overline{Y} \\ \mid \end{array} \leftrightarrow \begin{array}{c} X \mid Y \\ \overline{X} \mid \boxed{f} \\ \mid Z \end{array} .$$

Hence, for all simple $X, Y, Z \in \mathbf{C}$, we have $N_{X,Y}^Z = N_{Z,\overline{Y}}^X = N_{\overline{X},Z}^Y$.

Assumption 2.10. We will assume that our rigid C^* -tensor categories are countably generated, i.e., there is a countable set \mathcal{L} whose elements are self-dual objects in \mathbf{C} such that for each $Y \in \mathbf{C}$, there are $X_1, \dots, X_n \in \mathcal{L}$ such that

$$\mathbf{C}(X_1 \otimes \dots \otimes X_n, Y) \neq (0),$$

i.e., Y is (isomorphic to) a sub-object of $X_1 \otimes \dots \otimes X_n$.

For convenience, we will identify words on \mathcal{L} with their products, i.e., the word $\alpha = X_1 X_2 \dots X_n$ is identified with $X_1 \otimes X_2 \otimes \dots \otimes X_n$.

Definition 2.11. The fusion graph of \mathbf{C} with respect to $Y \in \mathbf{C}$, denoted $\mathcal{F}_{\mathbf{C}}(Y)$, is the oriented graph whose vertices are the isomorphism classes of simple objects of \mathbf{C} , and between simple objects $X, Z \in \mathbf{C}$, there are $N_{X,Y}^Z = \dim(\mathbf{C}(X \otimes Y \rightarrow Z))$

oriented edges pointing from X to Z . Note that if Y is self-dual, then by semi-simplicity, we have $N_{X,Y}^Z = N_{Z,Y}^X$, and we may ignore the orientation of the edges.

The fusion graph of \mathbf{C} with respect to \mathcal{L} (with \mathcal{L} as in Assumption 2.10), denoted $\mathcal{F}_{\mathbf{C}}(\mathcal{L})$, is the push out of the $\mathcal{F}_{\mathbf{C}}(Y)$ over the isomorphism classes of simple objects $Y \in \mathcal{L}$, i.e., the vertices are the same as before, and the edge set is the union of the edge sets of the $\mathcal{F}_{\mathbf{C}}(Y)$ for $Y \in \mathcal{L}$. If e is an edge in $\mathcal{F}_{\mathbf{C}}(\mathcal{L})$ which comes from an edge in $\mathcal{F}_{\mathbf{C}}(Y)$, then we color e by Y .

Since \mathcal{L} is countable, $\mathcal{F}_{\mathbf{C}}(\mathcal{L})$ has countably many vertices, although it may not be locally finite. However, $\mathcal{F}_{\mathbf{C}}(X)$ is always locally finite for $X \in \mathbf{C}$.

Given a vertex $[X]$ of $\mathcal{F}_{\mathbf{C}}(\mathcal{L})$, the number $\dim(X)$ is independent of the choice of representative of $[X]$. Again, we get a Frobenius-Perron weight vector on the vertices of $\mathcal{F}_{\mathbf{C}}(\mathcal{L})$, given by $(\dim(X))_{[X] \in V(\mathcal{F}_{\mathbf{C}}(\mathcal{L}))}$, which satisfies the following equation for each $Y \in \mathcal{L}$:

$$\dim(X) \dim(Y) = \sum_{[Z] \in V(\mathcal{F}_{\mathbf{C}}(Y))} N_{X,Y}^Z \dim(Z).$$

Remark 2.12. We require the objects of \mathcal{L} to be self-dual because we want a label in a string diagram to be consistent regardless of any critical points. We could allow ourselves to work with non-self-dual objects in \mathcal{L} at the cost of adding an orientation to our strands to keep track of the label after passing through a critical point.

Definition 2.13. To get a planar algebra $\mathcal{PA}(\mathbf{C})_{\bullet}$, for each word α on \mathcal{L} , let $\mathcal{PA}(\mathbf{C})_{\alpha} = \mathbf{C}(\alpha \rightarrow 1)$, whose elements are represented diagrammatically as

$$\begin{array}{c} | \alpha \\ \boxed{f} \end{array}$$

Frobenius reciprocity allows us to identify $\mathcal{PA}(\mathbf{C})_{\alpha}$ with $\mathbf{C}(\beta \rightarrow \gamma)$ where $\alpha = \beta\bar{\gamma}$:

$$\begin{array}{c} | \alpha \\ \boxed{f} \end{array} = \begin{array}{c} | \beta \\ \boxed{f} \\ | \gamma \end{array}.$$

We may now interpret any planar tangle in $\mathbb{P}_{\mathcal{L}}$ labelled by morphisms of \mathbf{C} as one morphism in \mathbf{C} in the usual way. First, isotope the tangle so that each string travels transversally to each horizontal line, except at finitely many critical points. Then isotope the tangle so that each labelled disk and each critical point occurs at a different vertical height, and read the diagram from bottom to top to see what the morphism is. The zig-zag relation, Frobenius-reciprocity, and pivotality ensure that the answer is well-defined.

2.3 From planar algebras to tensor categories

Given a fantastic planar algebra P_{\bullet} , we obtain its C^* -tensor category $\text{Pro}(P_{\bullet})$ of projections as described in [MPS10]. We briefly recall the construction here.

Definition 2.14. Let $\mathbf{Pro}(P_\bullet)$ (abbreviated \mathbf{Pro} when P_\bullet is understood) be the rigid C^* -tensor category given as follows.

Objects: The objects of \mathbf{Pro} are formal finite direct sums of projections in P_\bullet , i.e., all $p \in P_{\alpha \rightarrow \alpha}$ satisfying $p = p^2 = p^*$ for all words α on \mathcal{L} . The trivial object is the empty diagram.

Tensor: We tensor objects in \mathbf{Pro} by horizontal concatenation; e.g., if $p \in P_{\alpha \rightarrow \alpha}$ and $q \in P_{\beta \rightarrow \beta}$, then $p \otimes q \in P_{\alpha\beta \rightarrow \alpha\beta}$ is given by

$$\begin{array}{c} \alpha\beta \\ | \\ \boxed{p \otimes q} \\ | \\ \alpha\beta \end{array} = \begin{array}{c} \alpha \\ | \\ \boxed{p} \\ | \\ \alpha \end{array} \begin{array}{c} \beta \\ | \\ \boxed{q} \\ | \\ \beta \end{array} \in P_{\alpha\beta \rightarrow \alpha\beta}.$$

Note that the simple objects in \mathbf{Pro} are the simple projections in P_\bullet .

We extend the tensor product to direct sums of projections linearly.

Morphisms: The morphisms in \mathbf{Pro} are matrices of intertwiners between the projections. If $p \in P_{\alpha \rightarrow \alpha}$ and $q \in P_{\beta \rightarrow \beta}$, then elements in $\mathbf{Pro}(p, q) = qP_{\alpha \rightarrow \beta}p$ are all $x \in P_{\alpha \rightarrow \beta}$ such that $x = qxp$, i.e.,

$$\begin{array}{c} \alpha \\ | \\ \boxed{x} \\ | \\ \beta \end{array} = \begin{array}{c} \alpha \\ | \\ \boxed{p} \\ | \\ \alpha \\ | \\ \boxed{x} \\ | \\ \beta \\ | \\ \boxed{q} \\ | \\ \beta \end{array}.$$

We compose morphisms by vertical concatenation of elements in the planar algebra. If we have $x \in \mathbf{Pro}(p \rightarrow q)$ and $y \in \mathbf{Pro}(q \rightarrow r)$, then the composite xy is given by

$$\begin{array}{c} | \\ | \\ \boxed{xy} \\ | \end{array} = \begin{array}{c} | \\ | \\ \boxed{x} \\ | \\ \boxed{y} \\ | \end{array} \in \mathbf{Pro}(p \rightarrow r).$$

Composition of matrices of morphisms occurs in the usual way.

Tensoring: We tensor morphisms by horizontal concatenation. If $x \in \mathbf{Pro}(p_1 \rightarrow q_1)$ and $y \in \mathbf{Pro}(p_2 \rightarrow q_2)$, then the tensor product $x \otimes y$ is given by

$$\begin{array}{c} | \\ | \\ \boxed{x \otimes y} \\ | \end{array} = \begin{array}{c} | \\ | \\ \boxed{x} \\ | \end{array} \begin{array}{c} | \\ | \\ \boxed{y} \\ | \end{array}.$$

The tensor product of matrices of intertwiners is the tensor product of matrices followed by tensoring of morphisms.

Duality: The *duality* operation on objects and morphisms is rotation by π

$$\boxed{\bar{p}} = \begin{array}{c} \text{---} \\ | \\ \boxed{p} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \boxed{p} \\ | \\ \text{---} \end{array}.$$

The evaluation and coevaluation maps are given by the caps and cups between the projections in the obvious way.

Adjoint: The *adjoint* operation in \mathbf{Pro} is the identity on objects. The adjoint of a 1-morphism is the same as the adjoint operation in the planar algebra P_\bullet . If $x \in \mathbf{Pro}(p \rightarrow q)$ where $p \in P_{\alpha \rightarrow \alpha}$ and $q \in P_{\beta \rightarrow \beta}$, then consider $x \in P_{\alpha \rightarrow \beta}$, take the adjoint, which is an element in $P_{\beta \rightarrow \alpha}$, and consider the result x^* as an element in $\mathbf{Pro}(q \rightarrow p)$.

For matrices of intertwiners, the adjoint is the $*$ -transpose.

Example 2.15. We copy the example from [MPS10] as it is highly instructional. If $p, q \in P_{\alpha \rightarrow \alpha}$ are orthogonal, then if we define the matrix

$$u = \begin{pmatrix} p & q \end{pmatrix} \in \mathbf{Pro}(\text{id}_\alpha \oplus \text{id}_\alpha \rightarrow \text{id}_\alpha),$$

we get an isomorphism $p \oplus q = u^* u \simeq u u^* = p + q$.

Remark 2.16. Note that \mathbf{Pro} is strict. For any projection $p \in P_{\alpha \rightarrow \alpha}$, $p \otimes 1_{\mathbf{Pro}} = 1_{\mathbf{Pro}} \otimes p = p$ since $1_{\mathbf{Pro}}$ is the empty diagram. For all projections $p, q, r \in P_\bullet$,

$$(p \otimes q) \otimes r = \begin{array}{c} \text{---} \\ | \\ \boxed{p} \quad \boxed{q} \quad \boxed{r} \\ | \\ \text{---} \end{array} = p \otimes (q \otimes r).$$

The following theorem is well-known to experts, and one can easily work it out from the definitions. See part (ii) of the remark on page 10 of [Yam12] for more details.

Theorem 2.17.

- (1) Let \mathcal{C} be a strict rigid C^* -tensor category. Then $\mathbf{Pro}(\mathcal{PA}(\mathcal{C})_\bullet)$ is equivalent to \mathcal{C} .
- (2) Let P_\bullet be a fantastic planar algebra. Then $\mathcal{PA}(\mathbf{Pro}(P_\bullet))_\bullet = P_\bullet$.

Corollary 2.18. Suppose that

- $\mathcal{C} = \mathbf{Pro}(P_\bullet)$ and P_\bullet has string labels \mathcal{L} , or
- $P_\bullet = \mathcal{PA}(\mathcal{C})_\bullet$ and \mathcal{C} has countable generating set \mathcal{L} of self-dual objects.

Then we may identify the fusion graph Γ of P_\bullet with the fusion graph $\mathcal{F}_{\mathcal{C}}(\mathcal{L})$ of \mathcal{C} .

3 GJS results for fantastic planar algebras

Given a subfactor planar algebra P_\bullet , GJSW constructed a subfactor $N \subset M$ whose planar algebra is P_\bullet [GJS10, JSW10]. Moreover, they identified the factors as interpolated free group factors [GJS11].

Given a fantastic planar algebra P_\bullet associated to a C^* -tensor category \mathcal{C} , we mimic the construction of GJSW to obtain a factor M_0 and C^* -tensor categories \mathbf{Bim} and \mathbf{CF} of bifinite bimodules over M_0 such that \mathcal{C} is equivalent to \mathbf{Bim} and \mathbf{CF} .

Remark 3.1. Recall that when we suppress the \star of a input disk in the tape of a rectangle, it is assumed that \star is in the lower-left corner. Recall that if a string is labelled by the word $\alpha \in \Lambda$, it is read either top to bottom or left to right.

Assumption 3.2. For convenience, we assume that for each $c \in \mathcal{L}$, $\delta_c > 1$.

3.1 The graded algebras and their orthogonalized pictures

To start, we set $\text{Gr}_0(P) = \bigoplus_{\alpha \in \Lambda} P_\alpha$ where Λ denotes the set of all finite sequences of colorings for strings and endow $\text{Gr}_0(P_\bullet)$ with a multiplication \wedge which satisfies

$$x \wedge y = \begin{array}{|c|} \hline \alpha \quad \beta \\ \hline \begin{array}{|c|} \hline x \\ \hline \end{array} \begin{array}{|c|} \hline y \\ \hline \end{array} \\ \hline \end{array}$$

where $x \in P_\alpha$ and $y \in P_\beta$. We endow $\text{Gr}_0(P)$ with the following trace:

$$\text{tr}(x) = \begin{array}{|c|} \hline \Sigma CTL \\ \hline \alpha \\ \hline \begin{array}{|c|} \hline x \\ \hline \end{array} \\ \hline \end{array} \quad (1)$$

where $x \in P_\alpha$ and ΣCTL denotes the sum of all colored Temperley-Lieb diagrams, i.e. all planar ways of pairing the colors on top of x in a way which respects the word α .

Lemma 3.3. *The inner product on $\text{Gr}_0(P_\bullet)$ given by $\langle x, y \rangle = \text{tr}(y^* x)$ is positive definite. Furthermore, left and right multiplication by elements in $\text{Gr}_0(P_\bullet)$ is bounded with respect to this inner product*

The proof of the above lemma will closely follow the orthogonalization approach in [JSW10]. To begin, we define a new algebraic structure \star on $\text{Gr}_0(P_\bullet)$ defined as follows. Suppose $x \in P_\alpha$ and $y \in P_\beta$. Then by letting $|\alpha|$ denote the length of α , we have

$$x \star y = \sum_{\substack{\gamma \text{ s.t.} \\ \alpha = \alpha' \gamma \\ \beta = \bar{\gamma} \beta'}} \alpha' \begin{array}{|c|} \hline \gamma \\ \hline \begin{array}{|c|} \hline x \\ \hline \end{array} \begin{array}{|c|} \hline y \\ \hline \end{array} \\ \hline \end{array} \beta'$$

where it is understood that if a string connects two different colors, then that term in the sum is zero. We let $F_0(P_\bullet)$ be the vector space $\text{Gr}_0(P_\bullet)$ endowed with the

multiplication \star . Given x in $F_0(P_\bullet)$, let x_\emptyset denote the component of x in $P_\emptyset \cong \mathbb{C}$. We define a trace tr_F on $F_0(P_\bullet)$ by $\text{tr}_F(x) = x_\emptyset$. Since P_\bullet is a C^* -planar algebra, the sesquilinear form

$$\langle x, y \rangle = \text{tr}_{F_0(P_\bullet)}(x \star y^*) = \boxed{x} \text{---} \boxed{y^*}$$

is a positive definite inner product.

Set $\text{Epi}(CTL)$ to be the set of colored Temperley-Lieb boxes with strings at the top and bottom where any string touching the top of the box must be through. One can argue exactly as in section 5 of [JSW10] that the map $\Phi : \text{Gr}_0(P_\bullet) \rightarrow F_0(P_\bullet)$ given by

$$\Phi(x) = \sum_{E \in \text{Epi}(CTL)} \begin{array}{c} \boxed{E} \\ \boxed{x} \end{array}$$

is a bijection with the property that $\Phi(x \wedge y) = \Phi(x) \star \Phi(y)$, $\Phi(x^*) = \Phi(x)^*$ and $\text{tr}(x) = \text{tr}_F(\Phi(x))$. Hence \star is an associative multiplication, $F_0(P_\bullet)$ and $\text{Gr}_0(P_\bullet)$ are isomorphic as $*$ -algebras, and the inner product on $\text{Gr}_0(P_\bullet)$ is positive definite.

We now prove that left multiplication by $x \in F_0(P_\bullet)$ is bounded (this will closely follow arguments in [Bro12]). We may assume $x \in P_\alpha$ for a fixed word α . For fixed words β and γ such that $\alpha = \bar{\beta}\gamma$, the element

$$\begin{array}{c} \gamma \\ \boxed{x^*} \end{array} \begin{array}{c} \beta \\ \boxed{x} \end{array} \begin{array}{c} \gamma \end{array}$$

is positive in the finite dimensional C^* algebra $P_{\bar{\gamma} \rightarrow \bar{\gamma}}$, since for any $y \in P_{\bar{\gamma} \rightarrow \bar{\gamma}}$,

$$\langle x^* x \cdot y, y \rangle_{P_{\bar{\gamma} \rightarrow \bar{\gamma}}} = \begin{array}{c} \star \\ \boxed{y^*} \end{array} \begin{array}{c} \bar{\gamma} \\ \boxed{y} \end{array} \begin{array}{c} \star \\ \boxed{x^*} \end{array} \begin{array}{c} \beta \\ \boxed{x} \end{array} \begin{array}{c} \gamma \end{array} = \left\| \begin{array}{c} \bar{\beta} \\ \boxed{x} \end{array} \begin{array}{c} \gamma \\ \boxed{y} \end{array} \right\|_{L^2(F_0(P_\bullet))}^2 \geq 0.$$

Given x and w with $x \in P_\alpha$ with $\alpha = \bar{\beta}\gamma$, $x \star w$ is a sum of terms of the form

$$\begin{array}{c} \bar{\beta} \\ \boxed{x} \end{array} \begin{array}{c} \gamma \\ \boxed{w} \end{array} \begin{array}{c} \gamma \end{array},$$

and we see that the 2-norm of the above diagram is

$$\begin{array}{c} \star \\ \boxed{w^*} \end{array} \begin{array}{c} \star \\ \boxed{w} \end{array} \begin{array}{c} \gamma \\ \boxed{x^*} \end{array} \begin{array}{c} \beta \\ \boxed{x} \end{array} \begin{array}{c} \gamma \end{array} = \text{tr}(x^* x w w^*) \leq \|x^* x\|_{P_{\bar{\gamma} \rightarrow \bar{\gamma}}} \|w w^*\|_{L^2(F_0(P_\bullet))} \quad (2)$$

where $\|\cdot\|_{P_{\bar{\gamma} \rightarrow \bar{\gamma}}}$ is the operator norm in the C^* -algebra $P_{\bar{\gamma} \rightarrow \bar{\gamma}}$. Hence using Equation (2) repeatedly, we have

$$\|x \star w\|_{L^2(F_0(P))} \leq \left(\sum_{\alpha=\beta\gamma} \|x\|_{P_{\bar{\gamma} \rightarrow \bar{\gamma}}} \right) \cdot \|w\|_{L^2(F_0(P_\bullet))},$$

and thus left multiplication is bounded on $F_0(P_\bullet)$. The boundedness of right multiplication is similar.

Since the multiplication is bounded, we can represent $F_0(P_\bullet)$ on $L^2(F_0(P_\bullet))$ acting by left multiplication. We denote $M_0 = F_0(P_\bullet)''$. We also use M_0 to denote $\text{Gr}_0(P_\bullet)''$ acting on $L^2(\text{Gr}_0(P_\bullet))$, but it will be clear from context which picture we are using. Of course, from the discussion above, both von Neumann algebras are isomorphic.

Given $\alpha \in \Lambda$, we draw a blue string for a string labelled α . We will provide the α label only when it is possible to confuse α and $\bar{\alpha}$.

We define the graded algebra $\text{Gr}_\alpha(P_\bullet) = \bigoplus_{\beta \in \Lambda} P_{\bar{\alpha}\beta\alpha}$ with multiplication \wedge_α by

$$x \wedge_\alpha y = \begin{array}{c} \beta \quad \gamma \\ | \quad | \\ \boxed{x} \text{---} \boxed{y} \end{array}$$

for $x \in P_{\bar{\alpha}\beta\alpha}$ and $y \in P_{\bar{\alpha}\gamma\alpha}$, and trace

$$\text{tr}(x) = \frac{1}{\delta^\alpha} \begin{array}{c} \boxed{\Sigma CTL} \\ | \beta \\ \alpha \boxed{x} \alpha \\ \text{---} \bar{\alpha} \end{array}.$$

Note 3.4. Be warned that the multiplication \wedge_α in the GJSW diagrams when restricted to $P_{\bar{\alpha} \rightarrow \bar{\alpha}}$ is in the opposite order with the multiplication in the introduction!

The inner product

$$\langle \cdot, \cdot \rangle_{F_\alpha(P_\bullet)} = \begin{array}{c} \alpha \quad \alpha \\ \text{---} \boxed{} \text{---} \boxed{} \text{---} \end{array}$$

makes $P_{\bar{\alpha}\beta\alpha} \perp P_{\bar{\alpha}\gamma\alpha}$ for $\beta \neq \gamma$. We get a trace on $F_\alpha(P_\bullet)$ by $\text{tr}_{F_\alpha(P_\bullet)}(x) = \langle x, 1_\alpha \rangle_{F_\alpha(P_\bullet)}$ where 1_α is the horizontal strand labelled α .

The multiplication \star_α given by

$$x \star_\alpha y = \sum_{\substack{\kappa \text{ s.t.} \\ \beta = \beta' \kappa \\ \gamma = \bar{\kappa} \gamma'}} \begin{array}{c} \beta' \quad \kappa \quad \gamma' \\ | \quad \text{---} \quad | \\ \boxed{x} \text{---} \boxed{y} \end{array}$$

for $x \in P_{\bar{\alpha}\beta\alpha}$ and $y \in P_{\bar{\alpha}\gamma\alpha}$ makes $F_\alpha(P)$ isomorphic as a $*$ -algebra to $\text{Gr}_\alpha(P_\bullet)$, preserving the inner product. The same techniques as above with heavier notation

show that the inner product on $F_\alpha(P_\bullet)$ (hence $\text{Gr}_\alpha(P_\bullet)$) is positive definite and that left and right multiplication in $F_\alpha(P_\bullet)$ (hence $\text{Gr}_\alpha(P_\bullet)$) is bounded. We can therefore form the von Neumann algebra $M_\alpha = (F_\alpha(P_\bullet))''$ acting by left multiplication on $L^2(F_\alpha(P_\bullet))$ (or $(\text{Gr}_\alpha(P_\bullet))''$ acting by left multiplication on $L^2(\text{Gr}_\alpha(P_\bullet))$). Again, it will be clear from context which picture we are considering.

3.2 Factorality of M_α

In this section, we aim to prove the following theorem:

Theorem 3.5. *The algebra M_α is a II_1 factor. We have an embedding $\iota_\alpha : M_0 \hookrightarrow M_\alpha$ which is the extension of the map $F_0(P_\bullet) \rightarrow F_\alpha(P_\bullet)$ given by*

$$\iota_\alpha \left(\begin{array}{c} | \\ \boxed{x} \end{array} \right) = \begin{array}{c} | \\ \boxed{x} \\ \hline \end{array},$$

and $\iota_\alpha(M_0)' \cap M_\alpha = P_{\bar{\alpha} \rightarrow \bar{\alpha}}^{\text{op}} \cong P_{\alpha \rightarrow \alpha}$.

Throughout this subsection, we use the orthogonal picture $M_\alpha = (F_\alpha(P_\bullet))''$. Pick a specific color, $c \in \mathcal{L}$, which we will denote by the color **green**, and let A be the abelian von Neumann subalgebra of M_α generated by the cup element

$$\cup_\alpha = \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array}.$$

We will obtain the factorality of M_α by first examining $L^2(M_\alpha)$ as an A – A bimodule.

To begin, assuming $|\gamma| \geq 2$, we set

$$V_\gamma = \left\{ x \in P_{\bar{\alpha}\gamma\alpha} : \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} = \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} = 0 \right\},$$

$V = \bigoplus_{\gamma \in \Lambda} V_\gamma$ and \mathcal{V} the A – A bimodule generated by V . For $b \in \mathcal{L}$, let \mathcal{V}_b be the A – A bimodule generated by elements of the form

$$\begin{array}{c} | \\ \boxed{x} \\ \text{---} \text{---} \text{---} \end{array},$$

and let W be the A – A bimodule generated by $P_{\alpha \rightarrow \alpha}$. We claim that we have the following decomposition:

Lemma 3.6. *As A – A bimodules, $L^2(M_\alpha) = W \oplus \mathcal{V} \oplus \bigoplus_{b \in \mathcal{L}} \mathcal{V}_b$.*

Proof. The proof is exactly the same as [Bro12] Proposition 2.1 except that we induct on the length of a word in Λ as opposed to the number of strands of a single color. \square

Notice that we can decompose V further as $V = V_{cc} \oplus V_{co} \oplus V_{oc} \oplus V_{oo}$. Here, V_{cc} consists of elements in V whose leftmost and rightmost strings are colored c , V_{co} consists of elements of V whose leftmost string is colored c and rightmost string is colored differently than c , V_{oc} consists of elements of V whose leftmost string is colored differently than c and whose rightmost string is colored c , and V_{oo} consists of the elements in V whose leftmost and rightmost strings are colored other than c . We set \mathcal{V}_{cc} the $A - A$ bimodule generated by V_{cc} and we define \mathcal{V}_{co} , \mathcal{V}_{oc} , and \mathcal{V}_{oo} analogously.

Let $(V_{cc})_n$ be the subspace of V_{cc} spanned by boxes with a word of length n on top and let $\{\zeta_{n,i}\}$ be an orthonormal basis for $(V_{cc})_n$. It straightforward to see that the set

$$\left\{ \zeta_{n,i}^{l,r} = \delta_c^{-(l+r)/2} \left[\begin{array}{c} \text{---} \overbrace{\quad \quad \quad}^l \quad \boxed{\zeta_{n,i}} \quad \underbrace{\quad \quad \quad}_r \text{---} \\ \text{---} \end{array} \right] : n, l, r \in \mathbb{N} \right\}$$

is an orthonormal basis for \mathcal{V}_{cc} . There are similar orthonormal bases for \mathcal{V}_{co} , \mathcal{V}_{oc} and \mathcal{V}_{oo} . Let B denote V_{cc} , V_{co} , V_{oc} , V_{oo} or $P_{\bar{a}b\alpha}$ for $b \neq c$ and let \mathcal{B} denote \mathcal{V}_{cc} , \mathcal{V}_{co} , \mathcal{V}_{oc} , \mathcal{V}_{oo} , or \mathcal{V}_b . Let π, ρ denote the left, right representation of M_0 on $L^2(M_0)$ respectively. We have the following lemma whose proof is straightforward:

Lemma 3.7. *Let $\phi : \mathcal{B} \rightarrow \ell^2(\mathbb{N}) \otimes B \otimes \ell^2(\mathbb{N})$ be defined on the orthonormal basis of \mathcal{B} by*

$$\phi \left(\delta_c^{-(l+r)/2} \left[\begin{array}{c} \text{---} \overbrace{\quad \quad \quad}^l \quad \boxed{v} \quad \underbrace{\quad \quad \quad}_r \text{---} \\ \text{---} \end{array} \right] \right) = \xi_l \otimes v \otimes \xi_r$$

with $v \in B_n$, where $\{\xi_i | i \in \mathbb{N}\}$ is the usual orthonormal basis of $\ell^2(\mathbb{N})$. Then ϕ extends to a unitary operator and we have the following representations of \mathcal{U}_α :

- If $\mathcal{B} = \mathcal{V}_{cc}$, \mathcal{V}_{co} , \mathcal{V}_{oc} or \mathcal{V}_{oo} then

$$\phi \cdot \pi \left(\frac{\mathcal{U}_\alpha - 1}{\delta_c^{1/2}} \right) \cdot \phi^* = x \otimes 1 \otimes 1 \text{ where}$$

x is $s + s^*$ if the top leftmost color in for boxes in B is c and $s + s^* - \delta_c^{-1/2} e_{\xi_0}$ if the this color differs from c . Similarly,

$$\phi \cdot \rho \left(\frac{\mathcal{U}_\alpha - 1}{\delta_c^{1/2}} \right) \cdot \phi^* = 1 \otimes 1 \otimes x \text{ where}$$

x is $s + s^*$ if the top rightmost color in for boxes in B is c and $s + s^* - \delta_c^{-1/2} e_{\xi_0}$ if the this color differs from c .

- If $\mathcal{B} = \mathcal{V}_b$ and $b \neq c$ then

$$\phi \cdot \pi \left(\frac{\mathcal{U}_\alpha - 1}{\delta_c^{1/2}} \right) \cdot \phi^* = (s + s^* - \delta_c^{-1/2} e_{\xi_0}) \otimes 1 \otimes 1 \text{ and}$$

$$\phi \cdot \rho \left(\frac{\mathcal{U}_\alpha - 1}{\delta_c^{1/2}} \right) \cdot \phi^* = 1 \otimes 1 \otimes (s + s^* - \delta_c^{-1/2} e_{\xi_0}).$$

- (See [Bro12]) If $\mathcal{B} = \mathcal{V}_c$ then there is a unitary (which is not ϕ) $v : \mathcal{V}_c \rightarrow \ell^2(\mathcal{N}) \otimes L^2(P_{\bar{\alpha}c\alpha}) \otimes \ell^2(\mathbb{N})$ such that

$$\begin{aligned} v \cdot \pi \left(\frac{\mathcal{U}_\alpha - 1}{\delta_c^{1/2}} \right) \cdot v^* &= (s + s^*) \otimes 1 \otimes 1 \text{ and} \\ v \cdot \rho \left(\frac{\mathcal{U}_\alpha - 1}{\delta_c^{1/2}} \right) \cdot v^* &= 1 \otimes 1 \otimes (s + s^*). \end{aligned}$$

Here, s is the unilateral shift operator on $\ell^2(\mathbb{N})$ and e_{ξ_0} is the orthogonal projection in $\mathcal{B}(\ell^2(\mathbb{N}))$ onto the one-dimensional space spanned by ξ_0 .

We now show that the operators $s + s^*$ and $s + s^* - \delta_c^{-1/2} e_{\xi_0}$ are unitary equivalent in $\mathcal{B}(\ell^2(\mathbb{N}))$. We begin with the following lemma

Lemma 3.8. *The spectra of $s + s^*$ and $y = s + s^* - \delta_c^{-1/2} e_{\xi_0}$ are the same.*

Proof. Since the operators differ by a finite rank operator, by the Weyl-von Neumann Theorem (see [Kat95] p.523), they have the same essential spectrum. The operator $s + s^*$ has essential spectrum $[-2, 2]$ and since the complement of the essential spectrum in the spectrum is an isolated set of eigenvalues, we just need to show that y has no eigenvalues outside of $[-2, 2]$ since y is self adjoint and must have real spectrum.

To this end, let $\lambda > 2$ and $\xi = \sum_j x_j \xi_j$. If ξ is an eigenvector of y with eigenvalue λ then we have the equations

$$x_1 = (\lambda + \delta_c^{-1/2}) x_0 \text{ and } \lambda x_{n+1} = x_n + x_{n+2}$$

for $n \geq 0$. The characteristic equation for this linear recurrence is $x^2 - \lambda x + 1 = 0$ which has roots

$$l = \frac{\lambda + \sqrt{\lambda^2 - 4}}{2} \text{ and } r = \frac{\lambda - \sqrt{\lambda^2 - 4}}{2}.$$

This implies the existence of constants C and D with $x_n = C \cdot l^n + D \cdot r^n$ for all n . Since the sequence x_n is ℓ^2 , $C = 0$ meaning $x_n = x_0 \cdot r^n$ for all n . However, this means $x_1 = (\lambda + 1/\sqrt{\delta_c})x_0$ and $x_1 = rx_0$ must both be satisfied. This can only happen if $x_0 = 0$ since $\lambda + 1/\sqrt{\delta_c} > 1$ and $r < 1$. This implies $\xi = 0$.

Now let $\lambda < -2$. With l and r as above we must have $x_n = C \cdot l^n + D \cdot r^n$. This time we must have $D = 0$ and we obtain the equation $x_n = x_0 l^n$. This gives us the two equations

$$x_1 = (\lambda + \delta_c^{-1/2}) x_0 \text{ and } x_1 = lx_0$$

which implies $(\lambda - \frac{1}{2}(\lambda + \sqrt{\lambda^2 - 4}) + \delta_c^{-1/2})x_0 = 0$. This forces x_0 to be 0 since by the choice of λ , $\lambda - \frac{1}{2}(\lambda + \sqrt{\lambda^2 - 4}) < -1$ and $|\delta_c^{-1/2}| < 1$. Therefore $\xi = 0$ and y has no eigenvectors. \square

For any self-adjoint operator $a \in B(\mathcal{H})$, we set $\sigma(a)$ to be the spectrum of a . Given $\xi \in \mathcal{H}$, a Radon measure μ_ξ on the real line is induced by the formula $\mu_\xi(f) = \langle f(a)\xi, \xi \rangle$ for any bounded continuous f . We set \mathcal{H}_{ac} the Hilbert space of vectors ξ where μ_ξ is absolutely continuous with respect to the Lebesgue measure, \mathcal{H}_{sc} the Hilbert space of vectors ξ where μ_ξ is singular with respect to the Lebesgue measure, and \mathcal{H}_{pp} the Hilbert space of vectors ξ where μ_ξ has purely atomic measure. We define the absolutely continuous spectrum of a as the spectrum of a on \mathcal{H}_{ac} and denote it as $\sigma_{ac}(a)$. We define $\sigma_{cc}(a)$ and $\sigma_{pp}(a)$ in a similar manner.

We say a has uniform multiplicity 1 if there is a measure μ on the spectrum of a and a unitary $w : \mathcal{H} \rightarrow L^2(\sigma(a), \mu)$ such that for any $f \in L^2(\sigma(a), \mu)$, $waw^*(f)(x) = xf(x)$. In this case $\sigma_{ac}(a)$, $\sigma_{cc}(a)$ and $\sigma_{pp}(a)$ form a partition of $\sigma(a)$. See [Kat95] for more details.

Lemma 3.9. *y is unitary equivalent to $s + s^*$.*

Proof. We first show that y has uniform multiplicity 1. To this end, consider the following sequence of polynomials:

$$\begin{aligned} p_0(x) &= 1 \\ p_1(x) &= x + \delta_c^{-1/2} \\ p_{n+2}(x) &= xp_{n+1} - p_n(x) \text{ for } n \geq 0 \end{aligned}$$

By induction, it is straightforward to check that $p_n(y)(\xi_0) = \xi_n$ and hence the map $w : \mathcal{H} \rightarrow L^2(\sigma(a), \mu_{\xi_0})$ given by $w(\xi_n) = p_n$ is a unitary satisfying $wyw^*(f)(x) = xf(x)$ for all $f \in L^2(\sigma(a), \mu_{\xi_0})$. By the Kato-Rosenblum theorem (see [Kat95] p.540), if a and b are self adjoint operators and b is trace class then the absolutely continuous parts of b and $a + b$ are unitary equivalent. This implies that the absolutely continuous parts of y and $s + s^*$ are unitary equivalent. The absolutely continuous spectrum of $s + s^*$ is $[-2, 2] = \sigma(s + s^*)$ so the same must hold for y . As was discussed in the previous lemma, y has no spectral values outside $[-2, 2]$ and from above, the spectrum of y is partitioned into σ_{ac} , σ_{cc} and σ_{pp} . Therefore $\sigma_{pp} = \sigma_{cc} = \emptyset$ and $\sigma_{ac} = \sigma$ showing the unitary equivalence. \square

Therefore by conjugating by a unitary, we may assume that $L^2(M_\alpha) \cong L^2(A) \oplus \ell^2(\mathbb{N}) \otimes \mathcal{H} \otimes \ell^2(\mathbb{N})$ for some Hilbert space \mathcal{H} where on the second summand, $\pi\left(\frac{\cup-1}{\sqrt{\delta_c}}\right)$ acts as $(s + s^*) \otimes 1 \otimes 1$ and $\rho\left(\frac{\cup-1}{\sqrt{\delta_c}}\right)$ acts as $1 \otimes 1 \otimes (s + s^*)$. The lemmas below follow [JSW10] but we supply proofs for the readers' convenience.

Lemma 3.10. $A' \cap M_\alpha = AP_{\bar{\alpha} \rightarrow \bar{\alpha}}^{\text{op}}$.

Proof. First, note that $AP_{\bar{\alpha} \rightarrow \bar{\alpha}}^{\text{op}} \subseteq A' \cap M_\alpha$ in the obvious way

where we need the “op” since multiplication in the GJSW picture happens in the opposite order (see Note 3.4).

If $A' \cap M_\alpha$ were larger than $AP_{\bar{\alpha} \rightarrow \bar{\alpha}}^{\text{op}}$ then by looking at the orthogonal compliment of $AP_{\bar{\alpha} \rightarrow \bar{\alpha}}^{\text{op}}$ in $L^2(M_\alpha)$, there is a nonzero vector $\xi \in \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ with $((s+s^*) \otimes 1)\xi = (1 \otimes s + s^*)\xi$. Viewing $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ as the Hilbert Schmidt operators on $\ell^2(\mathbb{N})$ this means $\xi(s+s^*) = (s+s^*)\xi$ which is impossible since $s+s^*$ has no eigenvalues. \square

We will realize M_0 as a unital subalgebra of M_α via the map ι_α .

Lemma 3.11. $M'_0 \cap M_\alpha = P_{\bar{\alpha} \rightarrow \bar{\alpha}}^{\text{op}} \cong P_{\alpha \rightarrow \alpha}$ as an algebra.

Proof. If $x \in M'_0 \cap M_\alpha$ then by Lemma 3.10, $x \in AP_{\bar{\alpha} \rightarrow \bar{\alpha}}^{\text{op}}$, so we write x as an ℓ^2 sum

$$x = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\delta_c^n}} \cdot x_n \star \boxed{\begin{array}{c} \text{---} \cdots \text{---} \\ \text{---} \end{array}},$$

where $x_n \in P_{\bar{\alpha} \rightarrow \bar{\alpha}}^{\text{op}}$ for all n . Consider the following elements of M_0 :

$$z = \boxed{\begin{array}{c} \text{---} \cdots \text{---} \\ \text{---} \end{array}}, l_n = \frac{1}{\sqrt{\delta_c^n}} \boxed{\begin{array}{c} \text{---} \cdots \text{---} \\ \text{---} \end{array}}, \text{ and } r_n = \frac{1}{\sqrt{\delta_c^n}} \boxed{\begin{array}{c} \text{---} \cdots \text{---} \\ \text{---} \end{array}}.$$

By a direct diagrammatic computation, $z \star x - x \star z$ is the orthogonal sum

$$\sum_{n=1}^{\infty} (x_n + \frac{1}{\sqrt{\delta_c}} \cdot x_{n+1}) \star (l_n - r_n).$$

For this to be zero, we must have $x_{n+1} = -\sqrt{\delta_c} \cdot x_n$ for $n \geq 1$. Since the sum for x must be ℓ^2 , this implies $x_n = 0$ for $n \geq 1$, i.e., $x \in P_{\bar{\alpha} \rightarrow \bar{\alpha}}^{\text{op}}$.

Obviously the rotation by 180 degrees gives the isomorphism $P_{\bar{\alpha} \rightarrow \bar{\alpha}}^{\text{op}} \cong P_{\alpha \rightarrow \alpha}$. \square

Proof of Theorem 3.5. If x were in the center of M_α then we know that $x \in P_{\bar{\alpha} \rightarrow \bar{\alpha}}^{\text{op}}$. The element x must commute with

$$\boxed{\begin{array}{c} \text{---} \cdots \text{---} \\ \text{---} \end{array}},$$

giving the equation

$$\boxed{\begin{array}{c} \text{---} \cdots \text{---} \\ \text{---} \end{array}} \cdot \boxed{x} = \boxed{x} \cdot \boxed{\begin{array}{c} \text{---} \cdots \text{---} \\ \text{---} \end{array}}.$$

Joining the leftmost strings to the top implies that x is a scalar multiple of the identity. \square

3.3 Jones' Towers Associated to M_0

If α and β are in Λ such that $\alpha = \beta\gamma$ with $\gamma \in \Lambda$ then we have a unital, trace-preserving inclusion $\iota : M_\beta \rightarrow M_\alpha$ given by

$$\iota(x) = \frac{\beta}{\gamma} \boxed{x} \frac{\beta}{\gamma}.$$

When we write $M_\beta \subset M_\alpha$ we mean that M_β is included into M_α in the manner described above. We have the following theorem:

Theorem 3.12. *The following is a Jones' tower of factors:*

$$M_0 \subset M_\alpha \subset M_{\alpha\bar{\alpha}} \subset \cdots \subset M_{(\alpha\bar{\alpha})^n \alpha} \subset M_{(\alpha\bar{\alpha})^{n+1}} \subset \cdots$$

Moreover $[M_\alpha : M_0] = \delta_\alpha^2$.

Proof. This proof closely follows Section 4 in [JSW10]. We will show that $M_{\alpha\bar{\alpha}}$ is the basic construction of $M_0 \subset M_\alpha$. The proof for higher steps in the tower is the same but with heavier notation. To begin, set

$$e_0 = \frac{1}{\delta_\alpha} \begin{array}{|c|} \hline \text{Diagram: A square with two blue arcs on the left and right sides, facing each other.} \\ \hline \end{array} \in P_{\alpha\bar{\alpha} \rightarrow \alpha\bar{\alpha}}^{op} \subset M_{\alpha\bar{\alpha}}$$

then e_0 is a projection. It is a straightforward diagrammatic computation to show that if $x \in M_\alpha$ then $e_0 x e_0 = E_{M_0}(x)$ with E_{M_0} the trace preserving conditional expectation.

We now claim that $(M_\alpha, e_0)''$ is a II_1 factor. Indeed, if y were in the center of $(M_\alpha, e_0)''$ it would have to commute with M_0 , implying that $y \in P_{\alpha\bar{\alpha} \rightarrow \alpha\bar{\alpha}}$. The element y also has to commute with M_α , in particular it has to commute with the element

$$\begin{array}{|c|} \hline \text{Diagram: A square with a blue arc on the top side and a blue arc on the bottom side, both facing left.} \\ \hline \end{array}.$$

This implies y must be of the form

$$\begin{array}{c} \text{Diagram: A square with a blue line above it and a blue line below it, both labeled with } \bar{\alpha} \text{ on the left and right. Inside the square is the letter } y. \end{array},$$

but commuting with e_0 forces this diagram to be a scalar multiple of the identity.

One then observes that if $z \in (M_\alpha, e_0)''$, then $z e_0 = (\delta^\alpha)^2 E_{M_\alpha}(z e_0) e_0$. This is done by realizing that as a von Neumann algebra, $(M_\alpha, e_0)''$ is generated by M_α and $M_\alpha e_0 M_\alpha$ and hence one can assume z is in either of these spaces. The equality then becomes a straightforward diagrammatic check. From this, one can deduce that the map $x \mapsto \delta^\alpha x e_0$ from M_α to $(M_\alpha, e_0)''$ is a surjective isometry intertwining E_{M_0} on $L^2(M_\alpha)$ and left multiplication by e_0 . From this, we deduce that $(M_\alpha, e_0)''$ is the basic construction of $M_0 \subset M_\alpha$ and evaluating $[M_\alpha : M_0]$ is a matter of calculating the trace of e_0 .

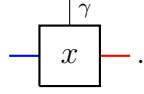
The same arguments applied to $M_\alpha \subset M_{\alpha\bar{\alpha}}$ implies that $[M_{\alpha\bar{\alpha}} : M_\alpha] = (\delta^\alpha)^2$ hence $[M_{\alpha\bar{\alpha}} : (M_\alpha, e_0)''] = 1$ i.e. $M_{\alpha\bar{\alpha}} = (M_\alpha, e_0)''$. \square

As an aside, since $M'_0 \cap M_\beta = P_{\beta \rightarrow \beta}$ for any β , it follows that the sequence of vector spaces $P_0, P_{\alpha \rightarrow \alpha}, P_{\alpha\bar{\alpha} \rightarrow \alpha\bar{\alpha}}, \cdots$ forms a subfactor planar algebra.

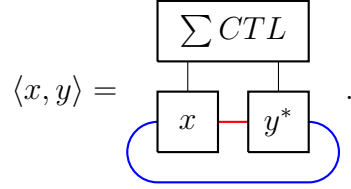
3.4 Some bifinite bimodules over M

Notation 3.13. We now use the notation M instead of M_0 as it makes the rest of this section easier on the eyes.

In this subsection and the next, we define a category \mathbb{C}_{bim} of bifinite bimodules over M . To begin, we use P_\bullet to construct another von Neumann algebra $\mathcal{M}_{\alpha,\beta}$ which contains the factors M_α and M_β as corners/cut-downs. Let $\text{Gr}_{\alpha,\beta}(P) = \bigoplus_{\gamma \in \Lambda} P_{\bar{\alpha}\gamma\beta}$. If $x \in P_{\bar{\alpha}\gamma\beta}$, we view x as



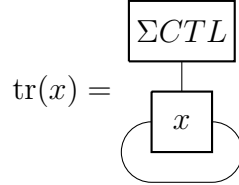
where we now draw **red** strings for strings labelled β . There is a sesquilinear form $\langle \cdot, \cdot \rangle$ on $\text{Gr}_{\alpha,\beta}(P)$ given by



As a vector space, $\text{Gr}_{\alpha,\alpha}(P_\bullet) = \text{Gr}_\alpha(P_\bullet)$. For $\alpha \neq \beta$, we form the $*$ -algebra $\mathcal{G}_{\alpha,\beta}$ which is generated by the vector spaces $\text{Gr}_\alpha(P_\bullet)$, $\text{Gr}_\beta(P_\bullet)$, $\text{Gr}_{\alpha,\beta}(P_\bullet)$ and $\text{Gr}_{\beta,\alpha}(P_\bullet)$, under the multiplication

$$\begin{array}{c} \gamma \\ | \\ \kappa \boxed{x} \end{array} \begin{array}{c} \theta \\ \text{---} \end{array} \wedge \begin{array}{c} \gamma' \\ | \\ \omega \boxed{x} \end{array} \begin{array}{c} \chi \\ \text{---} \end{array} = \delta_{\omega,\theta} \begin{array}{c} \gamma \\ | \\ \kappa \boxed{x} \end{array} \begin{array}{c} \theta \\ \text{---} \end{array} \begin{array}{c} \gamma' \\ | \\ \boxed{y^*} \end{array} \begin{array}{c} \chi \\ \text{---} \end{array},$$

where $\kappa, \theta, \omega, \chi \in \{\alpha, \beta\}$. There is also a (non-normalized) trace on $\mathcal{G}_{\alpha,\beta}$ given by



if $x \in \text{Gr}_\alpha(P_\bullet)$ or $\text{Gr}_\beta(P_\bullet)$, and is zero otherwise. Just as in the case for the algebras $\text{Gr}_\alpha(P_\bullet)$ one can show (by orthogonalizing) that the trace is positive definite and that multiplication is bounded. Therefore, one can form the von Neumann algebra $\mathcal{M}_{\alpha,\beta} = \mathcal{G}_{\alpha,\beta}''$ acting on $L^2(\mathcal{G}_{\alpha,\beta})$ by left and right multiplication. Set

$$p_\alpha = \begin{array}{|c|} \hline \alpha \\ \hline \end{array} \text{ and } p_\beta = \begin{array}{|c|} \hline \beta \\ \hline \end{array}.$$

We see that $p_\gamma \mathcal{M}_{\alpha,\beta} p_\gamma = M_\gamma$ (with the non-normalized trace) for $\gamma = \{\alpha, \beta\}$ so $L^2(\mathcal{M}_{\alpha,\beta})$ is naturally an $M_\alpha \oplus M_\beta$ bimodule. Hence we may consider $L^2(\mathcal{M}_{\alpha,\beta})$ as an $M - M$ bimodule via the embedding $x \mapsto \iota_\alpha(x) \oplus \iota_\beta(x) \in M_\alpha \oplus M_\beta$. Under this identification, we define $\mathcal{H}_{\alpha,\beta}$ to be the $M - M$ bimodule

$$\mathcal{H}_{\alpha,\beta} = p_\alpha \wedge L^2(\mathcal{M}_{\alpha,\beta}) \wedge p_\beta.$$

As mentioned above, we can give $\mathcal{G}_{\alpha,\beta}$ an orthogonalized inner product and multiplication exactly as in Section 3.1. We use the notation $\mathcal{F}_{\alpha,\beta}$ to denote the

vector space $\mathcal{G}_{\alpha,\beta}$ with the orthogonalized inner product and multiplication \star . The vector space $p_\alpha \star L^2(\mathcal{F}_{\alpha,\beta}) \star p_\beta$ is naturally an $M - M$ bimodule which is isomorphic to $p_\alpha \wedge L^2(\mathcal{M}_{\alpha,\beta}) \wedge p_\beta$. Our first lemma is proven by making use of the orthogonal picture:

Lemma 3.14. *The vector space of $M - M$ central vectors of $p_\alpha \star L^2(\mathcal{F}_{\alpha,\beta}) \star p_\beta$ is the vector space $P_{\bar{\alpha} \rightarrow \bar{\beta}}$ (rotating the GJS diagrams 90 degrees clockwise).*

Proof. As in Section 3.2, we let A be the von Neumann subalgebra of M generated by \cup and let W be the $A - A$ bimodule generated by $P_{\bar{\alpha} \rightarrow \bar{\beta}}$. With the same approach as 3.2, we see that as an $A - A$ bimodule, we have

$$\mathcal{H}_{\alpha,\beta} \cong W \bigoplus (\ell^2(\mathbb{N}) \otimes \mathcal{H} \otimes \ell^2(\mathbb{N}))$$

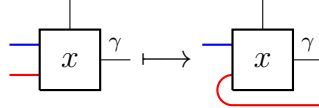
for \mathcal{H} an auxiliary Hilbert space. The operator \cup acts on the left of the second factor by $(s + s^*) \otimes 1 \otimes 1$ and on the right by $1 \otimes 1 \otimes (s + s^*)$. Therefore the proof of Lemma 3.10 applies here and we see that the $A - A$ central vectors of $L^2(\mathcal{F}_{\alpha,\beta})$ are exactly W . To finish the proof, one repeats the argument in Lemma 3.11. \square

Corollary 3.15. *The vector space of $M - M$ central vectors of $p_\alpha \wedge L^2(\mathcal{F}_{\alpha,\beta}) \wedge p_\beta$ is the vector space $P_{\bar{\alpha} \rightarrow \bar{\beta}}$ (rotating the GJS diagrams 90 degrees clockwise).*

For the rest of this section, we use the non-orthogonalized picture for $\mathcal{H}_{\alpha,\beta}$. We first write down some straightforward isomorphisms between these bimodules:

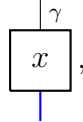
Lemma 3.16. *Let $\alpha, \beta, \gamma \in \Lambda$. Then as $M - M$ bimodules, $\mathcal{H}_{\alpha\beta,\gamma} \cong \mathcal{H}_{\alpha,\bar{\beta}\gamma}$.*

Proof. The map



is a unitary operator intertwining the left and right M -actions. \square

Definition 3.17. By Lemma 3.16, we now define the bimodule $H_\alpha := \mathcal{H}_{\alpha,\emptyset} \cong \mathcal{H}_{\emptyset,\bar{\alpha}}$. We draw elements of H_α with strings emanating from the bottom instead of the sides



and the left and right M -actions are given by the obvious diagrams. For $\xi \in p_\alpha \wedge \mathcal{M}_{\alpha,\emptyset} \wedge p_\emptyset \cong p_\emptyset \wedge \mathcal{M}_{\emptyset,\bar{\alpha}} \wedge p_{\bar{\alpha}}$ and $x, y \in M$,

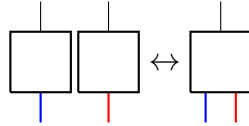
$$x\xi y = \begin{array}{c} \begin{array}{|c|} \hline x \\ \hline \end{array} \begin{array}{|c|} \hline \xi \\ \hline \end{array} \begin{array}{|c|} \hline y \\ \hline \end{array} \\ \text{blue wire from bottom of } \xi \end{array}.$$

The inner product on H_α is given by


$$\langle x, y \rangle_{H_\alpha} = \frac{\sum CTL}{\overline{y^*} \quad x} \quad (3)$$

where $\overline{y^*}$ is the rotation of y^* by 180 degrees.

Our goal will now be to show that $H_\alpha \otimes_M H_\beta \cong H_{\alpha\beta}$. In diagrams:



We will need the following lemma, whose purpose is to “close up” the space between the boxes.

Lemma 3.18. *The α -cup element  is positive and invertible in M .*

Proof. We induct on the size of α . If $|\alpha| = 1$ then it follows from [GJS10] that the operator is a free-poisson element whose spectrum is supported away from 0 hence the lemma holds for this case. Suppose that the lemma holds for some color pattern β and suppose $\alpha = \beta c$ for a fixed color c , denoted by a green string. It follows that the element

$$x = \text{Diagram: a square box with a red arc at the top and a green line at the bottom.}$$

is positive invertible in M_c . Let $z \in \mathcal{M}_{\emptyset, c}$ be the element

$$z = \text{Diagram: a square box with a green arc at the top and a green line at the bottom.}$$

Then the α -cup element has the form $z^* \wedge x \wedge z \in \mathcal{M}_{\emptyset, c}$ so it immediately follows that it is positive. Note that $z^* \wedge z$ is invertible so there is a positive constant k_z so that $\langle z^* \wedge z \wedge \xi, \xi \rangle \geq k_z \|\xi\|_2^2$ for all $\xi \in L^2(M)$. Letting r be the positive square root of x , there is a strictly positive constant k_r so that for all $\eta \in L^2(M_c)$, $\langle r \wedge \eta, r \wedge \eta \rangle \geq k_r \|\eta\|_2^2$. Therefore, for all $\xi \in L^2(M)$,

$$\begin{aligned} \langle z^* \wedge x \wedge z \wedge \xi, \xi \rangle &= \langle r \wedge z \wedge \xi, r \wedge z \wedge \xi \rangle \geq k_r \langle z \wedge \xi, z \wedge \xi \rangle \\ &= k_r \langle z^* \wedge z \wedge \xi, \xi \rangle \geq k_r k_z \|\xi\|_2^2 \end{aligned}$$

implying invertibility. □

Definition 3.19. Recall from [Con80, Bis97, Jon08] that given a bimodule ${}_M K_M$, a vector $\xi \in K$ is called right M -bounded if the map $L(\xi)^0: M \rightarrow K$ given by $m \mapsto \xi m$ extends to a bounded linear operator $L(\xi): L^2(M) \rightarrow K$. There is a similar definition of a left M -bounded vector, and if K is bifinite, then the sets of left and right M -bounded vectors agree. For such a bimodule K , we denote the set of left/right M -bounded vectors by $D(K)$.

Example 3.20. For all $\alpha, \beta \in \Lambda$, we have $D(H_{\alpha\beta}) = D(\mathcal{H}_{\alpha,\bar{\beta}}) = p_\alpha \wedge \mathcal{M}_{\alpha,\bar{\beta}} \wedge p_{\bar{\beta}}$.

Lemma 3.21. As $M - M$ bimodules, the Connes' fusion $H_\alpha \otimes_M H_\beta \cong H_{\alpha\beta}$.

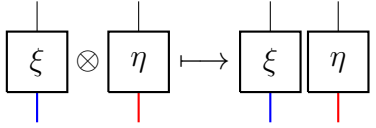
Proof. By [Bis97], $H_\alpha \otimes_M H_\beta$ is the completion of the algebraic tensor product $D(H_\alpha) \odot D(H_\beta)$ (first modding out by vectors of length zero) under the semi-definite inner product given by

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle_{H_\alpha \otimes_M H_\beta} = \langle \langle \xi_2 | \xi_1 \rangle_M \cdot \eta_1, \eta_2 \rangle_{H_\beta}$$

where $\langle \xi_2 | \xi_1 \rangle_M$ is the unique element in M satisfying

$$\langle \xi_1 x, \xi_2 \rangle_{H_\alpha} = \text{tr}(x \langle \xi_2 | \xi_1 \rangle_M) \text{ for all } x \in M.$$

We define a map u by the linear extension of

$$D(H_\alpha) \odot D(H_\beta) \longrightarrow \mathcal{H}_{\alpha,\bar{\beta}} \cong H_{\alpha\beta} \text{ by}$$


Obviously the map u intertwines the left and right M -actions and is M -middle linear. We will show it is isometric with dense range, and thus u has a unique extension to a $M - M$ bilinear unitary giving the desired isomorphism.

First, it is not hard to check directly that the M -valued inner product of the M -bounded vectors $\xi_1, \xi_2 \in D(H_\alpha)$ is given by

$$\langle \xi_2 | \xi_1 \rangle_M = \begin{array}{c} \boxed{\bar{\xi}_2^*} \quad \boxed{\xi_1} \\ \bar{\alpha} \quad \alpha \end{array}.$$

Hence if

$$\zeta = \sum_{i=1}^n \xi_i \otimes \eta_i \in D(H_\alpha) \odot D(H_\beta),$$

we immediately have

$$\|\zeta\|_2^2 = \sum_{i,j=1}^n \langle \langle \xi_j | \xi_i \rangle_M \cdot \eta_i, \eta_j \rangle_{H_\beta} = \begin{array}{c} \boxed{\sum CTL} \\ \boxed{\bar{\eta}_j^*} \quad \boxed{\bar{\xi}_j^*} \quad \boxed{\xi_i} \quad \boxed{\eta_i} \end{array}. \quad (4)$$

Since

$$u(\zeta) = \sum_{i=1}^n \begin{array}{c} \boxed{\xi_i} \quad \boxed{\eta_i} \\ \text{blue} \quad \text{red} \end{array},$$

we see that $\|u(\zeta)\|_2^2$ is the same diagram as in (4). Hence u can be extended to be an isometry on $H_\alpha \otimes_M H_\beta$.

For the rest of this proof we will only use the \wedge multiplication, so to clean up notation we will omit the \wedge . We must show the image of u is dense in $H_{\alpha\beta}$. We may assume $\delta_\alpha \leq \delta_\beta$, and the proof is similar if $\delta_\beta < \delta_\alpha$. Set

$$w = \begin{array}{|c|} \hline \text{ } \\ \hline \end{array}.$$

Then w^*w is positive and invertible in M , so if v is the polar part of w , then $v \in D(H_\alpha)$ with right support p_\emptyset and left support e under p_α . Choose partial isometries x_1, \dots, x_n in M_α such that x_1, \dots, x_{n-1} have right support e and orthogonal left supports. Choose the partial isometry x_n so that its left support is $1 - \sum_{i=1}^{n-1} x_i x_i^*$ and its right support is under e .

Similarly, there is a partial isometry $\tilde{v} \in D(H_\beta)$ whose right support is $f \leq p_{\bar{\beta}}$ and whose left support is p_\emptyset . Choose partial isometries $y_1, \dots, y_n \in M_{\bar{\beta}}$ such that y_1, \dots, y_{n-1} have left support f and orthogonal right supports. Choose the partial isometry y_n so its right support is orthogonal to $\sum_{i=1}^{n-1} y_i^* y_i$ and its left support is the right support of $x_n v \tilde{v}$ (we can do this since $\delta_\alpha \leq \delta_{\bar{\beta}}$).

It now follows that $z = \sum_{i=1}^n x_i v \tilde{v} y_i$ is a partial isometry in $D(H_{\alpha\beta})$ with full left support p_α . Note that every element, r in $D(H_{\alpha\beta})$ is of the form zx for $x \in M_{\bar{\beta}}$ since we simply choose $x = z^*r$. Since $zx = \sum_{i=1}^n (x_i v) \cdot (\tilde{v} y_i x)$ is in the image of u , we are finished. \square

Lemma 3.22. *As an $M - M$ bimodule, H_α is bifinite. Thus by Lemma 3.16, so is $\mathcal{H}_{\alpha,\beta}$.*

Proof. By the proof of Lemma 3.21, there is a partial isometry $v \in p_\alpha \wedge \mathcal{M}_{\alpha, \emptyset} \wedge p_\emptyset$ with right support p_\emptyset . The argument at the end of Lemma 3.21 shows that every element of $p_\alpha \wedge \mathcal{M}_{\alpha, \emptyset} \wedge p_\emptyset$ is of the form $x \wedge v$ for $x \in M_\alpha$. Therefore v is cyclic for the left action of M_α on $\mathcal{H}_{\alpha, \emptyset} \cong H_\alpha$, and since $[M_\alpha : M] < \infty$, $\dim_{M-}(H_\alpha) < \infty$. Similarly $\dim_{-M}(H_\alpha) < \infty$. \square

3.5 The categories $\text{Bim}(P_\bullet)$ and $\text{CF}(P_\bullet)$

We now define two rigid C^* -tensor categories $\text{Bim}(P_\bullet)$ and $\text{CF}(P_\bullet)$ whose objects are bifinite $M - M$ bimodules. We show given a fantastic planar algebra P_\bullet , we have equivalences $\text{Pro}(P_\bullet) \cong \text{Bim}(P_\bullet) \cong \text{CF}(P_\bullet)$.

Definition 3.23. If p is a projection in $P_{\alpha \rightarrow \alpha}$, we define $H_p = \mathcal{H}_{\emptyset, \bar{\alpha}} \wedge p$, which is a bifinite $M - M$ bimodule. Note that elements of H_p are obtained from elements in H_α by putting a p on the bottom. Thus the linear span of elements of the form



where $x \in p_\emptyset \wedge \mathcal{M}_{\emptyset, \bar{\alpha}} \wedge p$ forms a dense subset of H_p . Recall that elements of M act on the left and right as in the non-orthogonal picture. These actions clearly do not affect the p on the bottom.

Definition 3.24. Let $\mathbf{Bim}(P_\bullet)$ (abbreviated \mathbf{Bim}) be the strict rigid C^* -tensor category defined as follows.

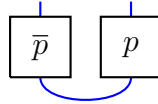
Objects: The objects of \mathbf{Bim} are finite direct sums of the bimodules H_p for the projections $p \in P_{\alpha \rightarrow \alpha}$ for $\alpha \in \Lambda$. Note that the unit object $1_{\mathbf{Bim}} = H_\emptyset \cong L^2(M)$.

Tensor: For $p \in P_{\alpha \rightarrow \alpha}$ and $q \in P_{\beta \rightarrow \beta}$, we define $H_p \otimes_{\mathbf{Bim}} H_q = H_{p \otimes q}$ where $p \otimes q$ is the tensor product in P_\bullet . The tensor product is extended to direct sums linearly. Note that since the tensor product of projections in P_\bullet is strict, so is the tensor product in \mathbf{Bim} .

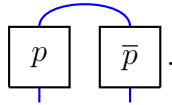
Morphisms: For $p \in P_{\alpha \rightarrow \alpha}$ and $q \in P_{\beta \rightarrow \beta}$, we define $\mathbf{Bim}(H_p \rightarrow H_q) = qP_{\alpha \rightarrow \beta}p$ and composition is the usual composition in P_\bullet . Morphisms between direct sums are matrices of such maps.

Tensoring: For $x_i \in \mathbf{Bim}(H_{p_i} \rightarrow H_{q_i}) = q_i P_{\alpha_i \rightarrow \beta_i} p_i$ for $i = 1, 2$, we define $x_1 \otimes_{\mathbf{Bim}} x_2$ as the tensor product of morphisms in P_\bullet . Similarly for matrices of such maps.

Duality: The dual of H_p is $H_{\bar{p}}$. The evaluation map $\text{ev}_{H_p}: H_{\bar{p}} \otimes_{\mathbf{Bim}} H_p \rightarrow H_\emptyset = 1_{\mathbf{Bim}}$ is given by the α -cup with projections \bar{p}, p on top:



Of course, the coevaluation map $1_{\mathbf{Bim}} \rightarrow H_p \otimes H_{\bar{p}}$ is given by the adjoint of ev_{H_p} , which is the α -cap with projections underneath:



One easily checks that the necessary relations hold.

The dual map is extended to direct sums linearly.

Adjoint: The adjoint map $*$ is the identity on all objects, and the adjoint of a morphism $x \in \mathbf{Bim}(H_p \rightarrow H_q) = qP_{\alpha \rightarrow \beta}p$ is the adjoint in the planar algebra $x^* \in pP_{\beta \rightarrow \alpha}q = \mathbf{Bim}(H_q \rightarrow H_p)$. The adjoint of a matrix of maps is the $*$ -transpose of the matrix.

Theorem 3.25. *The map $\mathbf{Pro} \rightarrow \mathbf{Bim}$ by $p \mapsto H_p$ and the identity on morphisms is an equivalence of categories.*

Proof. Note that $\mathbf{Pro}(p \rightarrow q) = qP_{\alpha \rightarrow \beta}p = \mathbf{Bim}(H_p \rightarrow H_q)$. One now checks that the described map is an additive, monoidal, dual-preserving, $*$ -preserving, fully faithful, essentially surjective functor. \square

Definition 3.26. Let $\mathbf{CF}(P_\bullet)$ (abbreviated \mathbf{CF} , which stands for *Connes' fusion*) be the rigid C^* -tensor category defined as follows.

Objects: The objects of \mathbf{CF} are finite direct sums of the bimodules H_p as in \mathbf{Bim} .

Tensor: For bimodules $K, L \in \mathbf{CF}$, we define $K \otimes_{\mathbf{CF}} L = K \otimes_M L$, the Connes' fusion of II_1 -factor bimodules.

The associator $a_{\mathbf{CF}}$ is defined by restricting to M -bounded vectors as in [Bis97].

Morphisms: $\mathbf{CF}(K \rightarrow L)$ is the set of $M - M$ bilinear maps $K \rightarrow L$. Composition is the usual composition of linear maps.

Tensoring: For $M - M$ bilinear maps $\varphi_i: K_i \rightarrow L_i$ for $i = 1, 2$, we define $\varphi_1 \otimes_M \varphi_2$ by the Connes' fusion of intertwiners. If we have M -bounded vectors $\xi_i \in K_i$ for $i = 1, 2$, then the map $\xi_1 \otimes \xi_2 \mapsto \varphi_1(\xi_1) \otimes \varphi_2(\xi_2)$ is clearly M -middle linear and bounded, so it extends to a unique $M - M$ bilinear map.

Duality: The dual of K is the contragredient bimodule $\overline{K} = \{\bar{\xi} | \xi \in K\}$ where $\lambda\bar{\xi} + \bar{\eta} = \overline{\lambda\xi + \eta}$ for all $\lambda \in \mathbb{C}$ and $\eta, \xi \in K$, and the action is given by $a\bar{\xi}b = \overline{b^*\xi a^*}$. The evaluation map

$$\text{ev}_K: \overline{K} \otimes_M K \longrightarrow H_\emptyset = L^2(M) = 1_{\mathbf{CF}}$$

is the unique extension of the map $\bar{\xi} \otimes \eta \mapsto \langle \xi | \eta \rangle_M$ where ξ, η are M -bounded vectors in K . The coevaluation map coev_K is the unique map in $\mathbf{CF}(1 \rightarrow K \otimes_M \overline{K})$ corresponding to $\text{id}_K \in \mathbf{CF}(K \rightarrow K)$ under the natural isomorphism given by Frobenius reciprocity. For an explicit formula, just pick an orthonormal left M -basis $\{\zeta\} \subset K$ (e.g., see [Con80, Pen12]), and we have that

$$\text{coev}_K(1_M) = \sum_{\zeta} \zeta \otimes \bar{\zeta}$$

is M -central and independent of the choice of $\{\zeta\}$. The zig-zag relation is now given by

$$\sum_{\zeta} \zeta \langle \zeta | \xi \rangle_M = \xi$$

for all M -bounded $\xi \in K$.

Adjoint: The adjoint map $*$ is the identity on all objects and on a morphism $x \in \mathbf{CF}(K \rightarrow L)$ is the adjoint linear operator $x^* \in \mathbf{CF}(L \rightarrow K)$.

Note \mathbf{CF} is a rigid C^* -tensor category by well known properties of Connes' fusion (e.g., see [Bis97, Pen12]). It is now our task to prove the following theorem:

Theorem 3.27. Define a map $\Phi: \mathbf{Bim} \rightarrow \mathbf{CF}$ as follows. First, Φ is the identity on objects. Second, for a morphism $x \in qP_{\alpha \rightarrow \beta}p$, we get an $M - M$ bimodule map $\Phi_x: H_p \rightarrow H_q$ by

$$\begin{array}{c} \begin{array}{|c|} \hline \xi \\ \hline \end{array} \begin{array}{|c|} \hline p \\ \hline \end{array} \mapsto \begin{array}{|c|} \hline \xi \\ \hline p \\ \hline x \\ \hline \end{array} \in H_q, \end{array} \quad (5)$$

Finally, Φ is applied entry-wise to matrices over such morphisms.

The map Φ is an equivalence of categories $\mathbf{Bim} \simeq \mathbf{CF}$.

In the lemmas below, unless otherwise stated, p, q are projections in $P_{\alpha \rightarrow \alpha}, P_{\beta \rightarrow \beta}$ respectively.

Remark 3.28. Note that composition of the bimodule maps given by Equation (5) corresponds to the usual composition in P_{\bullet} , i.e., if we have $x \in qP_{\alpha \rightarrow \beta}p$ and $y \in rP_{\beta \rightarrow \gamma}q$, then $\Phi_y \circ \Phi_x = \Phi_{yx}: H_p \rightarrow H_q$, where

$$\begin{array}{|c|} \hline yx \\ \hline \end{array} = \begin{array}{|c|} \hline x \\ \hline y \\ \hline \end{array}.$$

It is obvious that $\Phi_p = \text{id}_{H_p}$, so Φ is a functor.

Lemma 3.29. For p, q projections in P_{\bullet} , the maps

$$\phi_{p,q}: H_p \otimes_M H_q \rightarrow H_p \otimes_{\mathbf{Bim}} H_q = H_{p \otimes q}$$

(where $p \otimes q$ is the tensor product in P_{\bullet}) given by the unique extension of

$$\begin{array}{ccc} \begin{array}{|c|} \hline \xi \\ \hline \end{array} \begin{array}{|c|} \hline \eta \\ \hline \end{array} & \otimes & \begin{array}{|c|} \hline \xi \\ \hline \end{array} \begin{array}{|c|} \hline \eta \\ \hline \end{array} \\ \begin{array}{|c|} \hline p \\ \hline \end{array} \begin{array}{|c|} \hline p \\ \hline \end{array} & \mapsto & \begin{array}{|c|} \hline \xi \\ \hline p \\ \hline \end{array} \begin{array}{|c|} \hline \eta \\ \hline p \\ \hline \end{array} \end{array}$$

for $\xi \in D(H_p)$ and $\eta \in D(H_q)$, are $M - M$ bilinear isomorphisms which satisfy associativity, i.e., the following diagram commutes:

$$\begin{array}{ccc} (H_p \otimes_M H_q) \otimes_M H_r & \xrightarrow{a_{\mathbf{CF}}} & H_p \otimes_M (H_q \otimes_M H_r) \\ \phi_{p,q} \otimes_M \text{id}_{H_r} \downarrow & & \downarrow \text{id}_{H_p} \otimes_M \phi_{q,r} \\ (H_p \otimes_{\mathbf{Bim}} H_q) \otimes_M H_r & & H_p \otimes_M (H_q \otimes_{\mathbf{Bim}} H_r) \\ \phi_{p \otimes q, r} \downarrow & & \downarrow \phi_{p,q} \otimes r \\ (H_p \otimes_{\mathbf{Bim}} H_q) \otimes_{\mathbf{Bim}} H_r & \xrightarrow{=} & H_p \otimes_{\mathbf{Bim}} (H_q \otimes_{\mathbf{Bim}} H_r). \end{array}$$

Proof. As in the proof of Lemma 3.21, the map for $\xi \in D(H_p), \eta \in D(H_q)$ is an isometry intertwining the left and right M actions. Since the linear span of elements of the form $x \wedge y$ with $x \in D(\mathcal{H}_{\alpha, \emptyset})$ and $y \in D(\mathcal{H}_{\emptyset, \bar{\beta}})$ is equal to $D(\mathcal{H}_{\alpha, \bar{\beta}}) \cong D(H_{\alpha\beta})$ by the proof of Lemma 3.21, the above map is also surjective.

Associativity follows from looking at M -bounded vectors (see [Bis97]). \square

Lemma 3.30. *As complex vector spaces,*

$$\mathbf{CF}(H_p \rightarrow H_q) \cong q(P_{\alpha \rightarrow \beta})p = \mathbf{Bim}(H_p \rightarrow H_q).$$

Moreover, the composition of maps $\varphi \in \mathbf{CF}(H_p \rightarrow H_q)$ and $\psi \in \mathbf{CF}(H_q \rightarrow H_r)$ corresponds to the composition in \mathbf{Bim} .

Proof. Recall that an element $x \in q(P_{\alpha \rightarrow \beta})p = \mathbf{Bim}(H_p \rightarrow H_q)$ gives a map $\Phi_x \in \mathbf{CF}(H_p \rightarrow H_q)$ as in Equation (5), and composition of morphisms is exactly multiplication in P_\bullet by Remark 3.28. Note further that $\mathbf{Bim}(H_p \rightarrow H_q)$ and $\mathbf{CF}(H_p \rightarrow H_q)$ are finite dimensional, so it remains to show they have the same dimension.

By Frobenius reciprocity, we have a natural isomorphism

$$\mathrm{Hom}_{M-M}(H_p \rightarrow H_q) \cong \mathrm{Hom}_{M-M}(1 \rightarrow \overline{H_p} \otimes H_q),$$

and the latter space is naturally identified with the $M-M$ central vectors in $\overline{H_p} \otimes H_q$. Note that

$$\overline{H_p} \otimes_M H_q \cong H_{\bar{p}} \otimes_M H_q \cong H_{\bar{p} \otimes q} \cong p \wedge \mathcal{H}_{\alpha, \bar{\beta}} \wedge q.$$

From Corollary 3.15, the set of central vectors in $p \wedge \mathcal{H}_{\alpha, \bar{\beta}} \wedge q$ is $q(P_{\alpha \rightarrow \beta})p$

$$p \wedge_{\alpha} \mathcal{H}_{\alpha, \bar{\beta}} \wedge_{\bar{\beta}} q \ni \begin{array}{c} \text{---} \boxed{p} \text{---} \boxed{x} \text{---} \boxed{q} \text{---} \end{array} \longleftrightarrow \begin{array}{c} \text{---} \boxed{p} \text{---} \\ \boxed{x} \\ \text{---} \boxed{q} \text{---} \end{array} \in q(P_{\alpha \rightarrow \beta})p,$$

proving $\dim(\mathbf{CF}(H_p \rightarrow H_q)) = \dim(\mathbf{Bim}(H_p \rightarrow H_q))$. \square

Lemma 3.31. *If $x \in \mathbf{Bim}(H_p \rightarrow H_q) = qP_{\alpha \rightarrow \beta}p$ and $\Phi_x \in \mathbf{CF}(H_p \rightarrow H_q)$ is as in Equation (5), then $\Phi_x^* = \Phi_{x^*}$.*

Proof. By Equation (3), if $\xi \in D(H_p)$ and $\eta \in D(H_q)$, then

$$\langle \Phi_x \xi, \eta \rangle_{H_q} = \begin{array}{c} \boxed{\sum CTL} \\ | \\ \boxed{\bar{\eta}^*} \quad \boxed{\xi} \\ | \quad | \\ \boxed{\bar{q}} \quad \boxed{p} \\ | \quad | \\ \text{red strand} \quad \boxed{x} \\ \text{blue strand} \end{array} = \begin{array}{c} \boxed{\sum CTL} \\ | \\ \boxed{\bar{\eta}^*} \quad \boxed{\xi} \\ | \quad | \\ \boxed{\bar{q}} \quad \boxed{p} \\ | \quad | \\ \boxed{\bar{x}} \quad \text{blue strand} \\ \text{red strand} \end{array} = \langle \xi, \Phi_{x^*} \eta \rangle_{H_p}$$

where \bar{x} is the 180 degree rotation of x . □

For the next lemma, recall that the conjugate Hilbert space of K , is the set of formal symbols $\{\bar{\xi} | \xi \in H_p\}$ such that $\lambda \bar{\xi} + \bar{\eta} = \overline{\lambda \xi + \eta}$ for all $\lambda \in \mathbb{C}$ and $\eta, \xi \in H_p$, together with left and right M -actions given by $x \bar{\xi} y = \overline{y^* \xi x^*}$ for $x, y \in M$.

Lemma 3.32. *For $p \in P_{\alpha \rightarrow \alpha}$, define $\psi_p: H_{\bar{p}} \rightarrow \overline{H_p}$ (where \bar{p} is the dual projection of p in P_\bullet) by the unique extension of the map*

$$\begin{array}{c} \boxed{x} \\ | \\ \boxed{\bar{p}} \end{array} \mapsto \begin{array}{c} \overline{\boxed{x^*}} \\ | \\ \boxed{p} \end{array}$$

for $\xi \in D(H_{\bar{p}})$ (the blue strand on the left is labelled $\bar{\alpha}$). Then ψ_p is an $M - M$ bilinear isomorphism such that the following diagrams commute:

$$\begin{array}{ccccc} H_{\bar{p}} \otimes_M H_{\bar{q}} & \xrightarrow{\psi_p \otimes_M \psi_q} & \overline{H_p} \otimes_M \overline{H_q} & \xrightarrow{\cong} & \overline{H_q} \otimes_M \overline{H_p} \\ \downarrow \phi_{\bar{p}, \bar{q}} & & & & \downarrow \overline{\phi_{q, p}} \\ H_{\bar{p} \otimes \bar{q}} & \xrightarrow{=} & H_{\bar{q} \otimes p} & \xrightarrow{\psi_p \otimes q} & \overline{H_{q \otimes p}} \end{array}$$

and

$$\begin{array}{ccc} H_{\bar{p}} & \xrightarrow{\psi_p} & \overline{H_p} \\ \downarrow = & & \downarrow \overline{\psi_p} \\ H_p & \xrightarrow{\cong} & \overline{\overline{H_p}} \end{array}$$

where we just write \cong for the obvious isomorphism.

Proof. That ψ_p is a well-defined $M - M$ bilinear isomorphism is trivial. Commutativity of the diagrams follows by looking at M -bounded vectors. □

Lemma 3.33. For $p \in P_{\alpha \rightarrow \alpha}$, $\Phi_{\text{ev}_{H_p}^{\text{Bim}}} = \text{ev}_{H_p}^{\text{CF}}$ and $\Phi_{\text{coev}_{H_p}^{\text{Bim}}} = \text{coev}_{H_p}^{\text{CF}}$. Hence Φ preserves the rigid structure, and $\overline{\Phi_x} = \Phi_{\bar{x}} \in \text{CF}(H_{\bar{q}} \rightarrow H_{\bar{p}})$ for all $x \in \text{Bim}(H_p \rightarrow H_q)$.

Proof. In diagrams, $\text{ev}_{H_p}^{\text{CF}}$ is given for M -bounded vectors by

$$\text{ev}_{H_p}^{\text{CF}}(\bar{x} \otimes y) = \begin{array}{c} \begin{array}{|c|} \hline \bar{x} \\ \hline \end{array} \otimes \begin{array}{|c|} \hline y \\ \hline \end{array} \\ \begin{array}{|c|} \hline \bar{p} \\ \hline \end{array} \quad \begin{array}{|c|} \hline p \\ \hline \end{array} \end{array} \mapsto \begin{array}{c} \begin{array}{|c|} \hline \bar{x} \\ \hline \end{array} \quad \begin{array}{|c|} \hline y \\ \hline \end{array} \\ \begin{array}{|c|} \hline \bar{p} \\ \hline \end{array} \quad \begin{array}{|c|} \hline p \\ \hline \end{array} \end{array} = \langle x|y \rangle_M.$$

Hence using the isomorphism

$$\overline{H_p} \otimes_M H_p \xrightarrow{\psi_p \otimes_M \text{id}_{H_p}} H_{\bar{p}} \otimes_M H_p \xrightarrow{\phi_{\bar{p}, p}} H_{\bar{p} \otimes p}$$

from Lemmas 3.29 and 3.32, we have that $\text{ev}_{H_p}^{\text{CF}} = \Phi_{\text{ev}_{H_p}^{\text{Bim}}}$. Since Φ is $*$ -preserving by Lemma 3.31, we must have that $\Phi_{\text{coev}_{H_p}^{\text{Bim}}} = \text{coev}_{H_p}^{\text{CF}}$. Thus

$$\text{coev}_{H_p}^{\text{CF}}(1_M) = \begin{array}{c} \begin{array}{|c|} \hline p \\ \hline \end{array} \quad \begin{array}{|c|} \hline \bar{p} \\ \hline \end{array} \end{array}.$$

It is now easy to check that $\overline{\Phi_x} = \Phi_{\bar{x}} \in \text{CF}(H_{\bar{q}} \rightarrow H_{\bar{p}})$ for all $x \in \text{Bim}(H_p \rightarrow H_q)$. \square

Proof of Theorem 3.27. We must show Φ is additive, monoidal, dual-preserving, $*$ -preserving, fully faithful, and essentially surjective.

Additivity on objects and essentially surjective come for free, since the objects are the same. Additivity on morphisms and fully faithful follows from Lemma 3.30. Monoidal follows from Lemma 3.29, $*$ -preserving follows from Lemmas 3.30 and 3.31, and dual-preserving follows from Lemmas 3.32 and 3.33. Note that the results of the previous five lemmas extend in the obvious ways to direct sums by looking at matrices over the maps $H_p \rightarrow H_q$. \square

4 The isomorphism class of M

Assumption 4.1. First, note that the isomorphism classes of \mathbb{C} form a countable set if \mathbb{C} is a countably generated rigid C^* -tensor category. Let \mathcal{L} consist of a set of representatives for the isomorphism classes of objects in \mathbb{C} with dimension strictly greater than 1. In particular, this means $\mathcal{F}_{\mathbb{C}}(\mathcal{L})$ is not locally finite, and each vertex has self loops, since objects of the form $1 \oplus X$ are in \mathcal{L} .

With this assumption, we prove in Section 4.3 that $M \cong L(\mathbb{F}_{\infty})$. In the case that \mathbb{C} has finitely many isomorphism classes of simple objects, i.e., \mathbb{C} is a unitary fusion

category, we can find a single self dual object X with $\dim(X) > 1$ that generates \mathcal{C} . We also explain in Remark 4.27 that if we choose $\mathcal{L} = \{X\}$, then $M \cong L(\mathbb{F}_t)$ with

$$t = 1 + \dim(\mathcal{C})(\dim(X) - 1),$$

similar to the result in [GJS11].

To begin, we describe in the next two sections a semifinite algebra associated to P_\bullet . Many of the ideas in these two sections mirror [GJS11], except that the semifinite algebra makes all box-spaces orthogonal. One also must be more careful since the fusion graph $\mathcal{F}_{\mathcal{C}}(\mathcal{L})$ is not bipartite.

4.1 A semifinite algebra associated to P_\bullet

Set $\mathcal{G}_\infty = \bigoplus_{\alpha, \beta \in \Lambda} \text{Gr}_{\alpha, \beta}(P_\bullet)$. We endow \mathcal{G}_∞ with the multiplication

$$\begin{array}{c} \gamma \\ | \\ \boxed{x} \end{array} \begin{array}{c} \theta \\ \hline \end{array} \wedge \begin{array}{c} \gamma' \\ | \\ \boxed{x} \end{array} \begin{array}{c} \chi \\ \hline \end{array} = \delta_{\omega, \theta} \begin{array}{c} \gamma \\ | \\ \boxed{x} \end{array} \begin{array}{c} \theta \\ \hline \end{array} \begin{array}{c} \gamma' \\ | \\ \boxed{y^*} \end{array} \begin{array}{c} \chi \\ \hline \end{array},$$

where $\kappa, \theta, \omega, \chi \in \Lambda$. There is a (semifinite) trace, Tr on \mathcal{G}_∞ given by

$$\text{Tr}(x) = \begin{array}{c} \boxed{\Sigma CTL} \\ | \\ \boxed{x} \end{array}$$

for $x \in \text{Gr}_\alpha$, and $\text{Tr}(x) = 0$ for $x \in \text{Gr}_{\alpha, \beta}$ with $\alpha \neq \beta$.

Definition 4.2. As in Section 3.1, one argues (by once again orthogonalizing) that Tr is positive definite on \mathcal{G}_∞ and multiplication is bounded on $L^2(\mathcal{G}_\infty)$. Therefore, we form the (semifinite) von Neumann algebra $\mathcal{M}_\infty = \mathcal{G}_\infty''$ acting on $L^2(\mathcal{G}_\infty, \text{Tr})$.

We also use the von Neumann subalgebra $\mathcal{A}_\infty \subset \mathcal{M}_\infty$ which is generated by all boxes in \mathcal{G}_∞ with no strings on top.

As in Section 3.5, let

$$p_\alpha = \begin{array}{c} \boxed{} \\ \hline \end{array}.$$

It is easy to see that $p_\alpha \wedge \mathcal{M}_\infty \wedge p_\alpha = M_\alpha$ so that all factors in the various Jones towers in Section 3.3 appear as cut-downs/corners of \mathcal{M}_∞ . We now record some lemmas about the structure of \mathcal{M}_∞ and \mathcal{A}_∞ .

Lemma 4.3. \mathcal{M}_∞ is a II_∞ factor.

Proof. Note that $p_\emptyset \wedge \mathcal{M}_\infty \wedge p_\emptyset = M$ is a II_1 factor and the trace of $1 = 1_{\mathcal{M}_\infty} = \sum_{\alpha \in \Lambda} p_\alpha$ (with convergence of the orthogonal sum in the strong operator topology) is infinite, so we only need to show that \mathcal{M}_∞ is a factor. To do so, we show that the

central support of p_\emptyset in \mathcal{M}_∞ is 1, which is enough since $p_\emptyset \wedge \mathcal{M}_\infty \wedge p_\emptyset$ is a factor. We let

$$w_\alpha = \boxed{\text{blue arc}},$$

and we let v_α be the polar part of w_α . As was discussed in the proof of Lemma 3.21, v_α induces an equivalence of projections between p_\emptyset and $e \leq p_\alpha$. Let z be the central support of p_\emptyset (and e). Since $p_\alpha \wedge \mathcal{M}_\infty \wedge p_\alpha$ is a factor, the central support of e must lie above p_α , so $z \geq p_\alpha$. Since this holds for all $\alpha \in \Lambda$, we have $z \geq 1$. Hence $z = 1$, and we are finished. \square

Corollary 4.4. *The factor M_α is a δ_α -amplification of M .*

Corollary 4.5. *The algebras $\mathcal{M}_{\alpha,\beta}$ are II_1 factors.*

Proof. $\mathcal{M}_{\alpha,\beta}$ is the compression of \mathcal{M}_∞ by the finite projection $p_\alpha + p_\beta$. \square

Lemma 4.6. *We have a direct sum decomposition*

$$\mathcal{A}_\infty = \bigoplus_{v \in V(\mathcal{F}_\mathbb{C}(\mathcal{L}))} \mathcal{A}_v$$

where the sum is over all vertices v in the fusion graph $\mathcal{F}_\mathbb{C}(\mathcal{L})$, and each \mathcal{A}_v is a type I_∞ factor. If p is a minimal projection in $P_{\alpha \rightarrow \alpha}$ whose equivalence class represents the vertex v , then $p \leq 1_{\mathcal{A}_v}$.

Proof. For each vertex v , choose a minimal projection $p_v \in P_{\alpha_v \rightarrow \alpha_v}$ whose equivalence class corresponds to the vertex v . We first see that

$$p_v \wedge \mathcal{A}_\infty \wedge p_v = p_{\alpha_v} P_{\alpha_v \rightarrow \alpha_v} p_{\alpha_v} \cong \mathbb{C}$$

so \mathcal{A}_∞ has minimal projections. Letting $p \in P_{\alpha \rightarrow \alpha}$ and $q \in P_{\beta \rightarrow \beta}$, then the observation that $p \wedge \mathcal{A}_\infty \wedge q = q P_{\alpha \rightarrow \beta} p$ implies p is equivalent to q in the planar algebra sense if and only if p is equivalent to q in \mathcal{A}_∞ . Let 1_v be the central support of p_v . Then by construction, $\mathcal{A}_v = 1_v \wedge \mathcal{A}_\infty \wedge 1_v$ is a type I factor which must be type I_∞ since there are infinitely many mutually orthogonal projections equivalent to p_v in P_\bullet . It is easy to see that by construction, if $v \neq w$ then $1_v \wedge 1_w = 0$, which implies that

$$\mathcal{A}_\infty = \bigoplus_{v \in V(\mathcal{F}_\mathbb{C}(\mathcal{L}))} \mathcal{A}_v \oplus \mathcal{B}$$

for some von Neumann algebra \mathcal{B} .

We claim that $\mathcal{B} = \{0\}$. Indeed we know that p_α can be written as an orthogonal sum of projections equivalent to some of the p_w 's. Since $1 = \sum_{\alpha \in \Lambda} p_\alpha$, this implies $\sum_v 1_v = 1$, and thus $\mathcal{B} = \{0\}$. \square

Remark 4.7. For the rest of this section, all multiplication will be the \wedge multiplication in the GJS picture. Hence for the rest of this section, we just write xy for $x \wedge y$ for convenience.

We now show that we can obtain \mathcal{M}_∞ from a base “building block” \mathcal{A}_∞ and various free “corner elements.” Fixing a color c , which we again represent by the color **green**, we define

$$X_c = \sum_{\substack{\alpha \in \Lambda \\ |\alpha| \in 2\mathbb{N}}} \left[\text{diagram 1} \right] + \left[\text{diagram 2} \right].$$

Remark 4.8. We note that this sum defines a bounded operator. Indeed, the individual terms in the sum are supported under the mutually orthogonal family of projections $\{p_{c\alpha} + p_\alpha : |\alpha| \in 2\mathbb{N}\}$ and each term in the sum has operator norm

$$\left\| \left[\text{diagram 1} \right] + \left[\text{diagram 2} \right] \right\|_\infty^{1/2} = \left\| \left[\text{diagram 1} \right] + \left[\text{diagram 2} \right] \right\|_\infty^{1/2}.$$

We also note the simple fact that for $|\alpha|$ even, $p_{c\alpha}X_cp_\alpha$ is the corner diagram

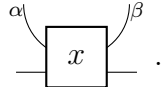


so each term in the sum defining X_c appears in the von Neumann algebra $W^*(\mathcal{A}_\infty, X_c)$. We only sum over α with $|\alpha|$ even as it makes computations involving freeness in Section 4.2 much easier.

The X_c elements give us a very nice way of obtaining \mathcal{M}_∞ from \mathcal{A}_∞ .

Lemma 4.9. $\mathcal{M}_\infty \cong W^*(\mathcal{A}_\infty, \{X_c : c \in \mathcal{L}\})$.

Proof. Every element in \mathcal{G}_∞ is a linear combination of elements of the following form:



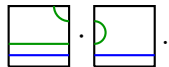
The above diagram is $x \in \mathcal{A}_\infty$ multiplied on the left and right by diagrams of the form



and their adjoints. This diagram is a product of diagrams of the form



so we only need to check that the above diagram is in $W^*(\mathcal{A}_\infty, \{X_c : c \in \mathcal{L}\})$ when $|\alpha|$ is odd. This is easy since it can be written as the product



□

There is a Tr -preserving conditional expectation $E : \mathcal{M}_\infty \rightarrow \mathcal{A}_\infty$ given by

$$E(x) = \begin{array}{c} \boxed{\Sigma CTL} \\ | \\ \boxed{x} \end{array},$$

and E induces normal completely positive maps $(\eta_{c,c} = \eta_c)_{c \in \mathcal{L}}$ on \mathcal{A}_∞ satisfying

$$\eta_c(y) = E(X_c y X_c) = \begin{array}{c} \text{---} \boxed{X_c} \text{---} \boxed{y} \text{---} \boxed{X_c} \text{---} \end{array}.$$

For $b \neq a$, we have trivial “off-diagonal” maps $\eta_{a,b}$ on \mathcal{A}_∞ satisfying $\eta_{a,b}(y) = E(X_a y X_b) = 0$. This gives a straightforward diagrammatic procedure for evaluating $E(y_0 X_{c_1} y_1 X_{c_2} \cdots X_{c_n} y_n)$ for $y_i \in \mathcal{A}_\infty$. First, write the word $y_0 X_{c_1} y_1 X_{c_2} \cdots X_{c_n} y_n$ as

$$\text{---} \boxed{y_0} \text{---} \boxed{X_{c_1}} \text{---} \boxed{y_1} \text{---} \cdots \text{---} \boxed{y_{n-1}} \text{---} \boxed{X_{c_n}} \text{---} \boxed{y_n} \text{---}.$$

Then sum over all planar ways to connect the strings on top. Whenever we see a term of the form

$$\text{---} \boxed{X_{c_i}} \text{---} \boxed{y} \text{---} \boxed{X_{c_j}} \text{---},$$

we replace it with $\eta_{c_i, c_j}(y)$. It is straightforward to check that E and the η_{c_i, c_j} satisfy the following recurrence relation:

$$\begin{aligned} E(y_0 X_{c_1} y_1 X_{c_2} \cdots y_{n-1} X_{c_n} y_n) \\ = \sum_{k=2}^n y_0 \cdot \eta_{c_1, c_k}(E(y_1 X_{c_2} \cdots X_{c_{k-1}} y_{k-1})) \cdot E(y_k X_{c_{k+1}} \cdots X_{c_n} y_n). \end{aligned} \quad (6)$$

Also, by definition of η_{c_i, c_j} , it follows that the map on $\mathcal{A}_\infty \otimes B(\mathcal{H})$ given by $(y_{i,j}) \mapsto (\eta_{c_i, c_j}(y_{i,j}))$ is normal and completely positive. This, combined with Recurrence (6) implies that the elements $(X_c)_{c \in \mathcal{L}}$ form an \mathcal{A}_∞ -valued semicircular family with covariance (η_{c_i, c_j}) as in [Shl99]. Since $\eta_{c_i, c_j} = 0$ for $c_i \neq c_j$, the family $(X_c)_{c \in \mathcal{L}}$ is free with amalgamation over \mathcal{A}_∞ with respect to E [Shl99]. We record what we have established above in the following lemma.

Lemma 4.10. $\mathcal{M}_\infty = W^*(\mathcal{A}_\infty, \{X_c : c \in \mathcal{L}\})$, and the elements $(X_c)_{c \in \mathcal{L}}$ form an \mathcal{A}_∞ -valued semicircular family with covariance (η_{c_i, c_j}) and are free with amalgamation over \mathcal{A}_∞ .

4.2 M as an amalgamated free product

By compressing the algebra \mathcal{A}_∞ by the projection $1_e = \sum_{|\alpha| \in 2\mathbb{N}} p_\alpha$, Lemma 4.6 implies that if we set $\mathcal{A}_e = W^*((P_{\beta \rightarrow \alpha})_{\alpha, \beta \in 2\mathbb{N}})$ then

$$\mathcal{A}_e = \bigoplus_{v \in V(\mathcal{F}_C(\mathcal{L}))} \mathcal{B}_v.$$

\mathcal{B}_v is a type I_∞ factor which is a cut-down of \mathcal{A}_v by 1_e . Note that every vertex in $v \in V(\mathcal{F}_C(\mathcal{L}))$ appears in the direct sum because every vertex possesses at least one self-loop.

Similarly, if one sets $1_o = \sum_{|\alpha| \in (1+2\mathbb{N})} p_\alpha$ and $\mathcal{A}_o = 1_o \mathcal{A}_\infty 1_o$, then we have

$$\mathcal{A}_o = \bigoplus_{v \in V(\mathcal{F}_C(\mathcal{L}))} \mathcal{C}_v$$

where \mathcal{C}_v is a type I_∞ factor which is a cut-down of \mathcal{A}_v by 1_o .

For each vertex $v \in \mathcal{F}_C(\mathcal{L})$, we choose a minimal projection $p_v \in \mathcal{A}_e$ whose equivalence class is represented by v with p_\emptyset the empty diagram. If $x \in \mathcal{A}_e$, it follows that

$$\eta_c(x) = \overline{\begin{array}{c} \boxed{x} \end{array}}$$

so that $\eta_c(p_v)$ is a finite projection in \mathcal{A}_o , and each η_c acts as a (non-unital) W^* algebra homomorphism from \mathcal{A}_e into \mathcal{A}_o . Set $Q = \sum_v p_v$. Then there is a family of partial isometries $(V_i)_{i \in I}$ satisfying $V_i^* V_i = Q$ and $\sum_{i \in I} V_i V_i^* = 1_{\mathcal{A}_e}$. Note that $\sum_{c \in \mathcal{L}} \eta_c(Q)$ defines a projection in \mathcal{A}_o since $\eta_a(Q) \perp \eta_b(Q)$ for $a \neq b$. We examine the compression of \mathcal{M}_∞ by $T = Q + \sum_{c \in \mathcal{L}} \eta_c(Q)$.

Lemma 4.11. $T\mathcal{M}_\infty T = W^*(T\mathcal{A}_\infty T, (TX_c T)_{x \in \mathcal{L}})$

Proof. By the choices of the partial isometries V_i , we know that if we let $W_i = \sum_{c \in \mathcal{L}} \eta_c(V_i)$ then $\sum_{i \in I} (V_i + W_i)(V_i + W_i)^* = 1$, so

$$T\mathcal{M}_\infty T = W^*(T\mathcal{A}_\infty T, \{(V_i + W_i)^* X_c (V_j + W_j) : i, j \in I, c \in \mathcal{L}\})$$

The relation $y \cdot X_c = X_c \eta_c(y)$ for any $y \in \mathcal{A}_e$ is a straightforward diagrammatic check. Applying this relation to each $(V_i + W_i)^* X_c (V_j + W_j)$ gives $TX_c T$. \square

Since $\sum_{i \in I} W_i W_i^* = 1_{\mathcal{A}_o}$, it follows that if $v \in V(\mathcal{F}_C(\mathcal{L}))$ then there is a minimal projection $q_v \in \mathcal{A}_o$ whose equivalence class represents v and sits under $1_{\mathcal{A}_o} \cdot T$. Note that this means that for all v , q_v and p_v are orthogonal minimal projections that sit under 1_v . We will next examine the cut-down of $T\mathcal{M}_\infty T$ by $F = \sum_v p_v + q_v$.

Suppose the vertices v and w are distinct and connected in $\mathcal{F}_C(\mathcal{L})$ and let $C(e)$ be the color of an edge connecting v and w . Let e_1, \dots, e_k be the (necessarily finite) collection of edges connecting v and w with $C(e_i) = c$. We let $u_{e_i}^v$ be a collection of partial isometries such that

$$(u_{e_i}^v)^* u_{e_j}^v = \delta_{ij} q_v \text{ and } \sum_{i=1}^k u_{e_i}^v (u_{e_i}^v)^* = \eta_c(p_w) \cdot 1_v$$

Let $r_v \in F\mathcal{A}_\infty F$ be a partial isometry satisfying $r_v r_v^* = p_v$ and $r_v^* r_v = q_v$ and $r_w \in F\mathcal{A}_\infty F$ be a partial isometry satisfying $r_w r_w^* = p_w$ and $r_w^* r_w = q_w$. The elements defined can be described diagrammatically as follows:

$$\begin{aligned} p_v &= \overline{\alpha_v} \boxed{p_v} \alpha_v, \quad p_w = \overline{\alpha_w} \boxed{p_w} \alpha_w, \quad q_v = \overline{\beta_v} \boxed{q_v} \beta_v, \quad q_w = \overline{\beta_w} \boxed{q_w} \beta_w, \\ u_{e_i}^v &= \overline{\alpha_w} \overline{\alpha_v} \boxed{u_{e_i}^v} \beta_v, \quad r_v = \overline{\alpha_v} \boxed{r_v} \beta_v, \quad r_w = \overline{\alpha_w} \boxed{r_w} \beta_w. \end{aligned}$$

Set x_i to be the following diagram:

$$x_i = \frac{\alpha_w}{\beta_v} \boxed{u_{e_i}^v}$$

We have the following lemma about the left and right supports of x_i :

Lemma 4.12. *The left support of x_i is p_w and the right support of x_i lies under the projection $1_w \cdot i_c(q_v)$ where we have for $z \in P_{\bar{\gamma} \rightarrow \bar{\chi}}$*

$$i_c(z) = \frac{\gamma}{\chi} \boxed{z}.$$

Furthermore if $i \neq j$ then the right supports of x_i and x_j are orthogonal.

Proof. Note that $x_i x_i^*$ is the diagram

$$\frac{\alpha_w}{\beta_v} \boxed{u_{e_i}^v} \frac{\beta_v}{\alpha_w} \boxed{(u_{e_i}^v)^*} \frac{\alpha_w}{\beta_v}.$$

For any $\gamma \in \Lambda$, let $\Phi : P_{\bar{\gamma}c \rightarrow \bar{\gamma}c}^{\text{op}} \rightarrow P_{\bar{\gamma} \rightarrow \bar{\gamma}}^{\text{op}}$ be the trace preserving conditional expectation where $P_{\bar{\gamma} \rightarrow \bar{\gamma}}^{\text{op}}$ includes unitally into $P_{\bar{\gamma}c \rightarrow \bar{\gamma}c}^{\text{op}}$ via i_c . The above diagram is a scalar multiple of $\Phi(u_{e_i}^v \cdot (u_{e_i}^v)^*)$. As $u_{e_i}^v \cdot (u_{e_i}^v)^*$ lies under $\eta_c(p_w) = i_c(p_w)$ it follows that this diagram is a scalar multiple of p_w , proving the claim about the left support. As left and right supports are equivalent projections, it follows that the right support of x_i lies under 1_w . We note that $\Phi(x_i^* x_i) = k(u_{e_i}^v)^* u_{e_i}^v = k q_v$ for some scalar k , implying that the right support of x_i must lie under $i_c(q_v)$. Finally, if $i \neq j$ then $x_i x_j^*$ is a constant multiple of $\Phi(u_{e_i}^v \cdot (u_{e_j}^v)^*)$. We know this must be a scalar multiple of p_w , but that scalar must be zero as $u_{e_i}^v \cdot (u_{e_j}^v)^*$ has trace zero. This proves the orthogonality of the right supports of the x_i . \square

Let f_i be the right support of x_i . As $i_c(p_v) \cdot 1_w$ can be written as a sum of k orthogonal projections equivalent to p_w , we conclude that $\sum_{i=1}^k f_i = i_c(p_v) \cdot 1_w$. We also recognize that $i_c(r_v)$ is a partial isometry with left support $i_c(q_v)$ and right support $i_c(p_v)$. We consider the elements $y_i = r_w^* \cdot x_i \cdot i_c(r_v^*)$ which diagrammatically look like

$$y_i = \frac{\beta_w}{\alpha_v} \boxed{r_w^*} \frac{\alpha_w}{\beta_v} \boxed{u_{e_i}^v} \frac{\beta_v}{\alpha_w} \boxed{r_v^*} \frac{\alpha_v}{\beta_w}$$

note that the left support of y_i is q_w . From Lemma 4.12, the right supports of the y_i are orthogonal and sum up to $i_c(p_v) \cdot 1_w$. As q_w is minimal in \mathcal{A}_∞ , it follows that y_i is a scalar multiple of a partial isometry. We define the partial isometry u_e^w by the equation $(u_{e_i}^w)^* = l \cdot y_i$ with l an appropriate constant. It follows that $(u_{e_i}^w)^* u_{e_j}^w = \delta_{ij} q_w$ and $\sum_{i=1}^k u_{e_i}^w (u_{e_i}^w)^* = \eta_c(p_v) \cdot 1_w$.

One can manipulate the diagram above to show that the relation

$$(u_e^v)^* = (\bar{l})^{-1} \cdot \frac{\beta_v}{\alpha_w} \boxed{r_v^*} \frac{\alpha_w}{\beta_v} \boxed{u_e^w} \frac{\beta_w}{\alpha_v} \boxed{r_w^*} \frac{\alpha_v}{\beta_w}$$

holds. This discussion thus proves the following useful lemma:

Lemma 4.13. *Let e_1, \dots, e_k be all of the edges of color c in the fusion graph connecting distinct vertices v and w . Then one can find partial isometries $\{u_{e_i}^v : 1 \leq i \leq k\}$ and $\{u_{e_i}^w : 1 \leq i \leq k\}$ in \mathcal{A}_o satisfying*

$$(u_{e_i}^v)^* u_{e_j}^v = \delta_{ij} q_v, \quad \sum_{i=1}^k u_{e_i}^v (u_{e_i}^v)^* = \eta_c(p_w) \cdot 1_v$$

$$(u_{e_i}^w)^* u_{e_j}^w = \delta_{ij} q_w, \quad \sum_{i=1}^k u_{e_i}^w (u_{e_i}^w)^* = \eta_c(p_v) \cdot 1_w$$

and the relations

$$(u_e^v)^* = (\bar{l})^{-1} \cdot \beta_v \boxed{r_v^*} \alpha_v \boxed{u_e^w} \beta_w \boxed{r_w^*} \alpha_w \quad \text{and} \quad (u_e^w)^* = l \cdot \beta_w \boxed{r_w^*} \alpha_w \boxed{u_e^v} \beta_v \boxed{r_v^*} \alpha_v$$

for some nonzero constant l .

We now turn our attention over to the edges e which are loops. Suppose v is a vertex and e_1, \dots, e_k represent all of the loops of color c connecting v to itself. As above, we can find partial isometries $u_{e_i}^v$ with right support q_v and orthogonal left supports under $1_v \cdot i_c(p_v)$, however for reasons that will become apparent later we desire a stronger property about these partial isometries.

Lemma 4.14. *There exists a set of partial isometries $\{u_{e_i}^v : 1 \leq i \leq k\}$ satisfying*

$$(u_{e_i}^v)^* u_{e_j}^v = \delta_{ij} q_v, \quad \sum_{i=1}^k u_{e_i}^v (u_{e_i}^v)^* = \eta_c(p_v) \cdot 1_v,$$

and the following relation:

$$(u_e^v)^* = k \cdot \beta_v \boxed{r_v^*} \alpha_v \boxed{u_e^v} \beta_v \boxed{r_v^*} \alpha_v$$

for k a unimodular constant.

Proof. Let $u_1 \in \mathcal{A}_\infty$ be a partial isometry with right support q_v and left support under $1_v \cdot i_c(p_v)$. Consider the operator y_1 satisfying

$$y_1^* = \beta_v \boxed{u_1^*} \alpha_v + \beta_v \boxed{r_v^*} \alpha_v \boxed{u_1} \beta_v \boxed{r_v^*} \alpha_v.$$

If this is zero, then by setting $k = -1$, we have produced a partial isometry satisfying the appropriate diagrammatic relation. If not, then y_1^* has left support q_v and right support under $1_v \cdot i_c(p_v)$ so it is a scalar multiple of a partial isometry. The operator y_1^* is also fixed under the map

$$x \mapsto \beta_v \boxed{r_v^*} \alpha_v \boxed{x^*} \beta_v \boxed{r_v^*} \alpha_v$$

so a scalar multiple of y_1 satisfies the appropriate diagrammatic relation and is a partial isometry. In either case, we have produced a partial isometry $u_{e_1}^v$ satisfying the diagrammatic relation. To produce another partial isometry $u_{e_2}^v$ with right support q_v , left support orthogonal to that of $u_{e_1}^v$, and satisfying the diagrammatic relation, we pick a partial isometry u_2 with right support q_v and left support orthogonal to that of $u_{e_1}^v$. By Lemma 4.12 and the discussion afterwards, the right supports of

$$\begin{array}{c} \beta_v \boxed{r_v^*} \alpha_v \boxed{u_2} \beta_v \boxed{r_v^*} \alpha_v \end{array} \quad \text{and} \quad k \cdot \begin{array}{c} \beta_v \boxed{r_v^*} \alpha_v \boxed{u_{e_1}^v} \beta_v \boxed{r_v^*} \alpha_v \end{array} = (u_{e_1}^v)^*$$

are orthogonal so it follows that by considering the element

$$y_2^* = \begin{array}{c} \beta_v \boxed{u_2^*} \alpha_v \end{array} + \begin{array}{c} \beta_v \boxed{r_v^*} \alpha_v \boxed{u_2} \beta_v \boxed{r_v^*} \alpha_v \end{array}.$$

and arguing as in the beginning of the proof, we produce the desired element $u_{e_2}^v$. Iterating this procedure produces the set $\{u_{e_i}^v : 1 \leq i \leq k\}$. \square

We therefore assume in the rest of the section that our partial isometries satisfy the relations in either Lemma 4.13 or Lemma 4.14. The identity $\sum_v \sum_{e \sim v} u_e^v (u_e^v)^* = \sum_{c \in \mathcal{L}} \eta_c(Q)$ follows by the choices of the partial isometries u_e^v . We therefore have the following lemma:

Lemma 4.15. *For each edge e of the fusion graph $\mathcal{F}_C(\mathcal{L})$, we define operators Y_e as follows: If $C(e) = c$ and e connects distinct vertices v and w then $Y_e = p_v X_c u_e^w + (u_e^w)^* X_c p_v + p_w X_c u_e^v + (u_e^v)^* X_c p_w$ and if e connects the vertex v to itself then $Y_e = p_v X_c u_e^v + (u_e^v)^* X_c p_v$. Then we have $F\mathcal{M}_\infty F = W^*(F\mathcal{A}_\infty F, (Y_e)_{e \in E})$*

Proof. First note that by compressing by the appropriate elements, of $F\mathcal{A}_\infty F$, each term in the sums defining Y_e is in $W^*(F\mathcal{A}_\infty F, (Y_e)_{e \in E(\mathcal{F}_C(\mathcal{L}))})$. The identity

$$\sum_{v \in V(\mathcal{F}_C(\mathcal{L}))} \sum_{\substack{e: s(e)=v \\ C(e)=c}} = p_v X_c u_e^{t(e)} (u_e^{t(e)})^* = T X_c T$$

implies that words in $T X_c T$ (compressed on either end by F) can be approximated by words in $p_v X_c u_e^w$ and their adjoints so by Lemma 4.11 we are done. \square

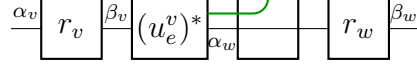
Lemma 4.16. *If v and w are distinct vertices connected by an edge e with $C(e) = c$ then $r_v (u_e^v)^* X_c p_w r_w$ and $p_v X_c (u_e^w)$ are nonzero scalar multiples of each other.*

If f is a loop at v then $r_v (u_f^v)^ X_c p_v r_v$ and $p_v X_c (u_f^v)$ are nonzero scalar multiples of each other.*

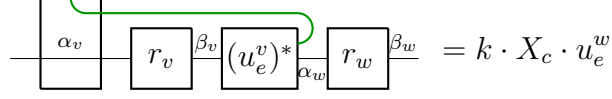
Proof. First observe that

$$p_v X_c u_e^w = X_c \eta_c(p_v) u_e^w = X_c u_e^w.$$

and $r_v(u_e^v)^* X_c p_w r_w = r_v(u_e^e)^* X_c r_w$. The element $r_v(u_e^e)^* X_c r_w$ is represented diagrammatically as



which can be rewritten as



for an appropriate constant $k \neq 0$, proving the first statement. The proof of the second statement uses the exact same diagrammatic argument. \square

We now examine the algebras

$$\mathcal{M}_e = W^*(F\mathcal{A}_\infty F, Y_e) = W^*(F\mathcal{A}_\infty F, p_v X_c u_e^w + (u_e^w)^* X_c p_v)$$

where the last equality follows from Lemma 4.16. First assume v and w are distinct and $\text{Tr}(p_v) \geq \text{Tr}(p_w) = \text{Tr}(q_w)$. One can show, using exactly the same techniques as in [GJS10] and [GJS11] that $p_v X_c u_e^w (u_e^w)^* X_c p_v$ is a free Poisson element with an atom of size $\frac{\text{Tr}(p_v) - \text{Tr}(p_w)}{\text{Tr}(p_v)}$ at 0 in the compressed algebra $p_v \mathcal{M}_\infty p_v$. Also, $(u_e^w)^* X_c p_v X_c u_e^w$ is a free poisson element with no atoms in the compressed algebra $q_w \mathcal{M}_\infty q_w$. Using the polar part of $p_v X_c u_e^w$ as well as the partial isometries r_v and r_w and Lemma 4.16, we see that

$$(1_v + 1_w) F \mathcal{M}_e F (1_v + 1_w) = (L(\mathbb{Z}) \otimes M_4(\mathbb{C})) \bigoplus_{\text{Tr}(p_v) - \text{Tr}(p_w)}^{p'_v, q'_v} M_2(\mathbb{C})$$

where the notation means that the copy of $M_2(\mathbb{C})$ has two orthogonal minimal projections p'_v and q'_v such that $p'_v \leq p_v$ and $q'_v \leq q_v$ and $\text{Tr}(p'_v) = \text{Tr}(p_v) - \text{Tr}(p_w)$.

If e is a loop at v then using the partial isometry r_v and Lemma 4.16, we see that

$$1_v \cdot F \mathcal{M}_e F \cdot 1_v = L(\mathbb{Z}) \otimes M_2(\mathbb{C}).$$

For simplicity, we set $X_e = p_v X_c u_e^w + (u_e^w)^* X_c p_v$ so that $\mathcal{M}_e = W^*(F \mathcal{M}_\infty F, X_e)$.

Lemma 4.17. *The algebras $(\mathcal{M}_e)_{e \in E(\mathcal{F}_C(\mathcal{L}))}$ are free with amalgamation over $F \mathcal{A}_\infty F$.*

Proof. From [Shl99] the elements X_e are $F \mathcal{A}_\infty F$ semicircular, so to show freeness we need only to show that if $e \neq e'$, $E(X_e y X_{e'}) = 0$ for all $y \in F \mathcal{A}_\infty F$. This can only be nonzero if e and e' have the same color, so we assume $C(e) = c = C(e')$. Also, this expectation can be nonzero only if e and e' share a common vertex, v .

We first assume that e and e' are not loops. Assume e connects v and w and e' connects v and w' . We see that $E(X_e p_v X_{e'}) = (u_e^w)^* i_c(p_v) u_{e'}^{w'} = (u_e^w)^* u_{e'}^{w'}$ which is zero since u_e^w and $u_{e'}^{w'}$ have orthogonal left supports if $w \neq w'$ and is clearly zero if $w = w'$. For $E(X_e q_w X_{e'})$ to be nonzero, we need $w = w'$. Assuming this, we get $p_v \Phi(u_e^w \cdot q_w \cdot (u_{e'}^{w'})^*) p_v$. As was discussed in the proof of Lemma 4.12,

$$\Phi(u_e^w \cdot q_w \cdot (u_{e'}^{w'})^*) = \Phi(u_e^w (u_{e'}^{w'})^*) = 0$$

since u_e^w and $u_{e'}^{w'}$ have orthogonal left supports. It is also straightforward to check that the only elements y in $F\mathcal{A}_\infty F$ where $X_e y X_{e'} \neq 0$ are of the form $y = a \cdot p_v + b \cdot q_w$.

We now assume that e is a loop at v and e' is an arbitrary edge connected to v . The discussion in the previous paragraph implies that $E(X_e p X_{e'}) = 0$ for any projection $p \in F\mathcal{A}_\infty F$; however, we also have to consider $E(X_e r_v^* X_{e'})$. This is the following diagram:

$$\begin{array}{c} \alpha_v \boxed{p_v} \alpha_v \boxed{u_e^v} \beta_v \boxed{r_v^*} \alpha_v \boxed{p_v} \alpha_v \boxed{u_{e'}^v} \beta_{e'}' = \alpha_v \boxed{r_v} \beta_v \boxed{r_v^*} \alpha_v \boxed{u_e^v} \beta_v \boxed{r_v^*} \alpha_v \boxed{u_{e'}^v} \beta_{e'}' \end{array},$$

where $\beta_{e'}' = \beta_v$ if e' is also a loop at v . By our choice of $u_{e'}^v$, this diagram is a scalar multiple of $r_v \cdot (u_e^v)^* u_{e'}^v$ which is 0. Similarly, $E(X_e r_v^* X_{e'}) = 0$ and we are done. \square

Note that we have

$$F\mathcal{M}_\infty F = \underset{F\mathcal{A}_\infty F}{*} (\mathcal{M}_e)_{e \in E(\mathcal{F}_C(\mathcal{L}))}.$$

The algebra $F\mathcal{A}_\infty F$ has the decomposition

$$F\mathcal{A}_\infty F = \bigoplus_{v \in \mathcal{F}_C(\mathcal{L})} \overset{p_v, q_v}{M_2(\mathbb{C})}.$$

Recall that $Q = \sum_v p_v$. Thus Q is an abelian projection with central support 1 in $F\mathcal{A}_\infty F$. Therefore (see for example [Dyk11] or [PS03])

$$Q\mathcal{M}_\infty Q = \underset{Q\mathcal{A}_\infty Q}{*} (Q\mathcal{M}_e Q)_{e \in E(\mathcal{F}_C(\mathcal{L}))}.$$

The algebra $Q\mathcal{A}_\infty Q$ is simply $\ell^\infty(V(\mathcal{F}_C(\mathcal{L})))$, the bounded functions on the vertices of $\mathcal{F}_C(\mathcal{L})$. If e connects two distinct vertices v and w with $\text{Tr}(p_v) \geq \text{Tr}(p_w)$ then

$$Q\mathcal{M}_e Q = L(\mathbb{Z}) \otimes M_2(\mathbb{C}) \bigoplus \overset{p_v'}{\mathbb{C}} \bigoplus \ell^\infty(V(\mathcal{F}_C(\mathcal{L})) \setminus \{v, w\})$$

and if e is a loop at v then

$$Q\mathcal{M}_e Q = L(\mathbb{Z}) \bigoplus \ell^\infty(V(\mathcal{F}_C(\mathcal{L})) \setminus \{v\}).$$

In the next section, we will use the free product expression for $Q\mathcal{M}_\infty Q$ to determine the isomorphism class of $M = p_\emptyset(Q\mathcal{M}_\infty Q)p_\emptyset$.

4.3 von Neumann algebras associated to graphs

The following notation will be useful in this section:

Notation 4.18. Throughout this section, we will be concerned with finite von Neumann algebras (\mathcal{N}, tr) which can be written in the form

$$\mathcal{N} = \underset{\gamma_0}{\mathcal{N}_0} \oplus \bigoplus_{j \in J} \overset{p_j}{L(\mathbb{F}_{t_j})} \oplus \bigoplus_{k \in K} \overset{q_k}{M_{n_k}} \underset{\alpha_k}{}.$$

where \mathcal{N}_0 is a diffuse hyperfinite von Neumann algebra, $L(\mathbb{F}_{t_j})$ is an interpolated free group factor with parameter t_j , M_{n_k} is the algebra of $n_k \times n_k$ matrices over the scalars, and the sets J and K are at most finite and countably infinite respectively. We use p_j to denote the projection in $L(\mathbb{F}_{t_j})$ corresponding to the identity of $L(\mathbb{F}_{t_j})$ and q_k to denote a minimal projection in M_{n_k} . The projections p_j and q_k have traces γ_j and α_k respectively. Let p_0 be the identity in \mathcal{N}_0 with trace γ_0 . We write $\overset{p,q}{M}_2$ to mean M_2 with a choice of minimal orthogonal projections p and q .

We begin by letting Γ be a connected weighted graph with weighting γ . Let $\ell^\infty(\Gamma)$ be the bounded functions on the vertices of Γ and let p_v be the delta function at v . We endow $\ell^\infty(\Gamma)$ with a trace Tr satisfying $\text{Tr}(p_v) = \gamma_v$. As in [Har12] we now describe how to use the edges to associate a free product von Neumann algebra $\mathcal{N}(\Gamma)$ to Γ .

Definition 4.19. Let e be an edge in Γ . We define algebras \mathcal{N}_e as follows: If e connects two distinct edges v and w with $\gamma_v \geq \gamma_w$ then

$$\mathcal{N}_e = M_2(\mathbb{C}) \otimes_{2\gamma_w} L(\mathbb{Z}) \oplus_{\gamma_v - \gamma_w} \overset{p_v^e}{\mathbb{C}} \oplus \ell^\infty(\Gamma \setminus \{v, w\}).$$

If e is a loop at the vertex v then

$$\mathcal{N}_e = \overset{p_v}{L(\mathbb{Z})}_{\gamma_v} \oplus \ell^\infty(\Gamma \setminus \{v\}).$$

If e connects two different vertices then the trace on $M_2(\mathbb{C}) \otimes_{2\gamma_w} L(\mathbb{Z}) \oplus_{\gamma_v - \gamma_w} \overset{p_v^e}{\mathbb{C}}$ is given by $\text{tr}_{M_2} \otimes \text{tr}_{L(\mathbb{Z})} + \text{tr}_{\mathbb{C}}$. \mathcal{N}_e includes $\ell^\infty \Gamma$ by letting $p_w = e_{1,1} \otimes 1$ and $p_v = e_{2,2} \otimes 1 + p_v^e$ so that $p_v^e \leq p_v$. If e is a loop that \mathcal{N}_e then \mathcal{N}_e includes $\ell^\infty(\Gamma)$ in the obvious way.

Let E_e be the Tr -preserving conditional expectation from \mathcal{N}_e to $\ell^\infty(\Gamma)$. We define $\mathcal{N}(\Gamma) = \overset{*}{\ell^\infty(\Gamma)} (\mathcal{N}_e, E_e)_{e \in E(\Gamma)}$.

Note that if Γ has no loops then this definition of $\mathcal{N}(\Gamma)$ is the same as $\mathcal{M}(\Gamma)$ in [Har12]. Also observe that $Q\mathcal{M}_\infty Q$ in section 4.2 is $\mathcal{N}(\mathcal{F}_\mathbb{C}(\mathcal{L}))$. We now define some notation which will be useful in determining the isomorphism class of $\mathcal{N}(\Gamma)$.

Definition 4.20. We write $v \sim w$ if v and w are connected by at least 1 edge in Γ and denote $n_{v,w}$ be the number of edges joining v and w . We set $\alpha_v^\Gamma = \sum_{w \sim v} n_{v,w} \gamma_w$, and define $B(\Gamma) = \{v \in V(\Gamma) : \gamma_v > \alpha_v^\Gamma\}$. Note that if there is a loop, e , at v then $v \notin B(\Gamma)$.

Assume for the moment that $\Gamma \subset \Gamma'$ are finite, connected, weighted graphs with at least two edges (so that $\mathcal{N}(\Gamma)$ and $\mathcal{N}(\Gamma')$ are finite von Neumann algebras). There is a natural inclusion $\mathcal{N}(\Gamma) \rightarrow \mathcal{N}(\Gamma')$ which will not be unital if Γ' has a larger vertex set. We will prove the following theorem, which is along the same lines as [Har12].

Theorem 4.21. $\mathcal{N}(\Gamma)$ has the form

$$\mathcal{N}(\Gamma) \cong L(\mathbb{F}_{t_\Gamma})^{p^\Gamma} \oplus \bigoplus_{v \in B(\Gamma)} \mathbb{C}^{r_v^\Gamma - \alpha_v^\Gamma}$$

where $r_v^\Gamma \leq p_v$ and t_Γ is such that this algebra has the appropriate free dimension. If Γ is a proper subgraph of Γ' then the unital inclusion $p^\Gamma \mathcal{N}(\Gamma) p^\Gamma \rightarrow p^\Gamma \mathcal{N}(\Gamma') p^\Gamma$ is a standard embedding of interpolated free group factors.

See [Dyk93] and [Dyk11] for a discussion of free dimension and rules for computing it. Also we refer the reader to [Dyk93] for the definition of a standard embedding of interpolated free group factors. Whenever A and B are interpolated free group factors and A is unitaly included into B then we write $A \xrightarrow{s.e.} B$ to indicate that the inclusion of A into B is a standard embedding. In this section, we will extensively use the following properties of standard embeddings which can be found in [Dyk93] and [Dyk95].

- (1) If A is an interpolated free group factor, the canonical inclusion $A \rightarrow A * \mathcal{N}$ is a standard embedding whenever \mathcal{N} is of the form in Notation 4.18.
- (2) A composite of standard embeddings is a standard embedding.
- (3) If $A_n = L(\mathbb{F}_{s_n})$ with $s_n < s_{n+1}$ for all n and $\phi_n : A_n \xrightarrow{s.e.} A_{n+1}$, then the inductive limit of the A_n with respect to the ϕ_n is $L(\mathbb{F}_s)$ where $s = \lim_{n \rightarrow \infty} s_n$.
- (4) If $t > s$ then $\phi : L(\mathbb{F}_s) \xrightarrow{s.e.} L(\mathbb{F}_t)$ if and only if for any nonzero projection $p \in L(\mathbb{F}_s)$, $\phi|_{pL(\mathbb{F}_s)p} : pL(\mathbb{F}_s)p \xrightarrow{s.e.} \phi(p)L(\mathbb{F}_t)\phi(p)$.

Theorem 4.21 was proved in [Har12] in the case that Γ contained no loops, so we only to need to modify the arguments there to incorporate what happens when Γ contains loops. We will prove Theorem 4.21 by inducting on the number of edges in the graph. We divide this into two lemmas.

Lemma 4.22. Suppose Γ is a connected graph with 2 edges, one of which is a loop. Then $\mathcal{N}(\Gamma)$ is of the form in Theorem 4.21.

Proof. There are two cases to consider. The first when Γ has one vertex with two loops and the other when Γ has two vertices, w and w with a loop at v and an edge connecting v to w .

Case 1: Assume Γ has one vertex with two loops. It follows immediately from the definition of $\mathcal{N}(\Gamma)$ that $\mathcal{N}(\Gamma) = L(\mathbb{F}_2)$ which is in agreement with Theorem 4.21.

Case 2: Assume Γ has two vertices w and w with a loop, e , at v and an edge, f , connecting v to w . There are two subcases to consider: when $\gamma_v \geq \gamma_w$ and $\gamma_v < \gamma_w$.

Case 2a: Assume $\gamma_v \geq \gamma_w$ and set $D = \ell^\infty(\Gamma)$. $\mathcal{N}(\Gamma)$ has the form

$$\mathcal{N}(\Gamma) = \left(L(\mathbb{Z}) \oplus_{\gamma_v}^{p_v} \oplus_{\gamma_w}^{p_w} \mathbb{C} \right) *_D \left(M_2 \otimes_{2\gamma_w} L(\mathbb{Z}) \oplus_{\gamma_v - \gamma_w}^{p_v^f} \mathbb{C} \right).$$

Note that the central support of p_v is 1 in \mathcal{N}_f . By [DR11],

$$p_v \mathcal{N}(\Gamma) p_v = L(\mathbb{Z}) * p_v \mathcal{N}_f p_v = L(\mathbb{Z}) * \left(L(\mathbb{Z}) \oplus_{\gamma_w}^{p_v} \oplus_{\gamma_v - \gamma_w}^{p_v^f} \mathbb{C} \right)$$

which is an interpolated free group factor $L(\mathbb{F}_t)$ for some appropriate t . By amplifying, it follows that $\mathcal{N}(\Gamma)$ is an interpolated free group factor, in agreement with Theorem 4.21.

Case 2b: Assume $\gamma_v < \gamma_w$. We note that the central support of p_v in \mathcal{N}_f is $1 - p_w^f$ and following the same algorithm as in Case 2a gives

$$p_v \mathcal{N}(\Gamma) p_v = L(\mathbb{Z}) * p_v \mathcal{N}_f p_v = L(\mathbb{Z}) * L(\mathbb{Z}) = L(\mathbb{F}_2),$$

so by amplifying, $\mathcal{N}(\Gamma) = L(\mathbb{F}_t) \oplus_{\gamma_w}^{p_w^f} \mathbb{C}$ for appropriate t . This agrees with Theorem 4.21. \square

Lemma 4.23. *Suppose $\Gamma \subset \Gamma'$ are finite weighted graphs and assume $\mathcal{N}(\Gamma)$ is of the form in Theorem 4.21. In addition, suppose Γ' is obtained from Γ by one of the following three operations:*

- (1) Γ and Γ' have the same edge set and Γ' is obtained from Γ by adding a loop e at a vertex v ,
- (2) Γ and Γ' have the same edge set and Γ' is obtained from Γ by adding an edge e between two distinct vertices v and w , or
- (3) Γ' is obtained from Γ by adding a vertex v and an edge e connecting $w \in V(\Gamma)$ to v .

Then $\mathcal{N}(\Gamma')$ is also of the form in Theorem 4.21 and $p^\Gamma \mathcal{N}(\Gamma) p^\Gamma \xrightarrow{s.e.} p^\Gamma \mathcal{N}(\Gamma') p^\Gamma$.

Proof. The steps in proving the lemma assuming operations (2) or (3) are exactly the same as in [Har12], so we only need to assume that Γ' is obtained from Γ via operation (1). Set $D = \ell^\infty(\Gamma) = \ell^\infty(\Gamma')$. We have $\mathcal{N}(\Gamma') = \mathcal{N}_e *_D \mathcal{N}(\Gamma)$ and by assumption,

$$p_v \mathcal{N}(\Gamma) p_v = L(\mathbb{F}_t) \oplus_{\gamma_v}^{p_v^\Gamma} \oplus_{\gamma_w}^{r_v^\Gamma} \mathbb{C}$$

where r_v^Γ can possibly be zero but $p_v^\Gamma = p_v \cdot p^\Gamma \neq 0$. By [DR11],

$$p_v \mathcal{N}(\Gamma') p_v = L(\mathbb{Z}) * p_v \mathcal{N}(\Gamma) p_v = L(\mathbb{Z}) * \left(L(\mathbb{F}_t) \oplus_{\gamma_v}^{p_v^\Gamma} \oplus_{\gamma_w}^{r_v^\Gamma} \mathbb{C} \right)$$

which is an interpolated free group factor. By amplification, it follows that $\mathcal{N}(\Gamma')$ is of the form in Theorem 4.21. By [Dyk93], The inclusion $p_v^\Gamma \mathcal{N}(\Gamma) p_v^\Gamma \rightarrow p_v^\Gamma \mathcal{N}(\Gamma') p_v^\Gamma$ is equivalent to the inclusion

$$L(\mathbb{F}_t) \rightarrow L(\mathbb{F}_t) * p_v^\Gamma \left(L(\mathbb{Z}) * \left(\mathbb{C}^{p_v^\Gamma} \oplus \mathbb{C}^{r_v^\Gamma} \right) \right) p_v^\Gamma$$

which is a standard embedding. As $p_v^\Gamma \leq p^\Gamma$ it follows that $p^\Gamma \mathcal{N}(\Gamma) p^\Gamma \xrightarrow{s.e.} p^\Gamma \mathcal{N}(\Gamma') p^\Gamma$. \square

We notice that Theorem 4.21 follows from Lemmas 4.22 and 4.23. Indeed, if $\Gamma \subset \Gamma'$ are finite, connected, weighted graphs then Γ' can be obtained from Γ by applying operations (1), (2), and (3) above. Also, the composite of standard embeddings is a standard embedding and standard embeddings are preserved by cut-downs.

Before determining the isomorphism $M \cong L(\mathbb{F}_\infty)$ we note the following lemma whose proof is a straightforward induction exercise using the algorithms in [DR11].

Lemma 4.24. *Suppose Γ consists of a vertex v connected to vertices w_1 through w_k by a total of n edges. Assume further that $\gamma_v \leq \gamma_{w_i}$ for all i . Then $p_v \mathcal{N}(\Gamma) p_v = L(\mathbb{F}_t)$ where $t \geq n$.*

Theorem 4.25. $M \cong L(\mathbb{F}_\infty)$.

Proof. Let Γ denote $\mathcal{F}_C(\mathcal{L})$ so that $M \cong p_\emptyset \mathcal{N}(\Gamma) p_\emptyset$. We build up Γ by an increasing union of finite connected subgraphs Γ_k , each of which contain the vertex $*$ so that M is the inductive limit of the algebras $p_\emptyset \mathcal{N}(\Gamma_k) p_\emptyset$. All vertices in Γ_k must have weight larger than 1 since for p an irreducible projection in P_\bullet , $p \otimes \bar{p}$ must have a subprojection equivalent to the trivial one. It follows that for k sufficiently large, $p_\emptyset \mathcal{N}(\Gamma_k) p_\emptyset$ is an interpolated free group factor and since $p_\emptyset \mathcal{N}(\Gamma_k) p_\emptyset \xrightarrow{s.e.} p_\emptyset \mathcal{N}(\Gamma_{k+1}) p_\emptyset$ we know that $M = p_\emptyset \mathcal{N}(\Gamma) p_\emptyset$ must be an interpolated free group factor. Let $\tilde{\Gamma}_n$ be the subgraph of Γ_n whose vertices are $*$ and v_1, \dots, v_k which are connected to $*$ in Γ_k and whose edges are exactly the edges connecting $*$ to each v_i to $*$ in Γ_n . Assuming there are k_n such vertices, we see from Lemma 4.24 that $p_\emptyset \mathcal{N}(\tilde{\Gamma}_n) p_\emptyset = L(\mathbb{F}_{r_n})$ for $r_n \geq k_n$. Since either $p_\emptyset \mathcal{N}(\tilde{\Gamma}_n) p_\emptyset = p_\emptyset \mathcal{N}(\Gamma_n) p_\emptyset$ or $p_\emptyset \mathcal{N}(\tilde{\Gamma}_n) p_\emptyset \xrightarrow{s.e.} p_\emptyset \mathcal{N}(\Gamma_n) p_\emptyset$ it follows that $p_\emptyset \mathcal{N}(\Gamma_n) p_\emptyset = L(\mathbb{F}_{t_n})$ for $t_n \geq r_n$. There are infinitely many edges emanating from $*$. Therefore k_n and hence t_n can be arbitrarily large so it follows that $M = L(\mathbb{F}_t)$ where $t = \lim_n t_n = \infty$. \square

Corollary 4.26. $M_\alpha \cong L(\mathbb{F}_\infty)$ for all $\alpha \in \Lambda$.

Proof. M_α is an amplification of M . \square

Remark 4.27. With a bit more careful analysis, using the techniques in [Har12], one can show that if \mathbb{C} has infinitely many simple objects then the factor M is $L(\mathbb{F}_\infty)$ no matter the choice of the generating set \mathcal{L} . When \mathbb{C} has finitely many simple objects then we can find t finite with $M \cong L(\mathbb{F}_t)$. To do this, one can take \mathcal{L} to consist of a single element $[X]$. Applying the analysis in section 4.2 shows that M is a cutdown of $\mathcal{N}(\Gamma)$ for Γ the fusion graph of \mathbb{C} with respect to the color $[X]$. Keeping track of the free dimension yields the isomorphism $M \cong L(\mathbb{F}(1 + \dim(\mathbb{C})(\dim(X) - 1)))$.

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